On the Definition of Transfer Factors*

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To Friedrich Hirzebruch on his sixtieth birthday

CONTENTS

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0. Introduction

This paper is concerned with a specific question in harmonic analysis on reductive groups over a local field. Central to it is the notion of an endoscopic group ([Sh; L_2], and for a perhaps definitive treatment [K-S]). This notion is still exotic and difficult to grasp, partly because its origins lie beyond the periphery of harmonic analysis, in the L-group (which first arose in the theory of Eisenstein series) and in the study of Shimura varieties, and partly because it still has not achieved in sufficient generality its original purpose, the analysis of the internal structure of L-packets of irreducible representations.

Roughly speaking, L-packets occur in the classification of the irreducible representations of the group $G(F)$ of F-valued points on a reductive group over the local field F because there are two types of conjugacy within $G(F)$, that realized by elements of $G(F)$ and that realized by elements of $G(\bar{F})$. Only the first appears when the harmonic analysis is treated from a strictly analytic viewpoint, but the second intervenes when the harmonic analysis is applied to problems in number theory, especially to the study of L-functions, and leads to a coarser classification of irreducible representations than equivalence. The coarse classes are called L-packets, and they are to be analyzed individually with the help of endoscopic groups.

An endoscopic group H is not a subgroup of G, but we can associate to a conjugacy class in $H(F)$ several conjugacy classes in $G(F)$, and the harmonic analysis on $G(F)$ is related to that on $H(F)$ by means of the transfer of orbital integrals. This refers to pairs of functions, one f on $G(F)$ and one f^H on $H(F)$ whose orbital integrals on associated conjugacy classes are related by transfer factors [see (1.4)].

The definition of transfer factors that not only allow one to attach to each f at least one f^H but also behave well with respect to functoriality has not been easy, and if it were not that they had been proved to exist over the real field [Sh], it would have been difficult to maintain confidence in the possibility of transfer or in the usefulness of endoscopy.

The contribution of this paper is not to prove the existence of the transfer, that is to attach to f at least one f^H, but simply to define the transfer factor, disentangling the conditions imposed or suggested by the harmonic analysis, by Galois cohomology, by the trace formula, and by the constructions over $\mathbf{R}$ to arrive at an explicit definition that clearly must be the transfer factor if it exists at all and that even over the real field is an improvement over the construction of [Sh] which was not sufficiently explicit. We flatter ourselves that this definition is an advance and that it is not merely our lack of skill that has made it so hard to come by, but the difficulty of the subject.

In Sect. 1 we are more explicit about the transfer of orbital integrals and transfer factors, recalling in particular the first example that was studied, the group $SL(2)$. Their definition appears in Sect. 3, where it will be seen that the transfer factor is the product of five terms, two, Δ_{II} and Δ_{IV}, serving to meet basic requirements of harmonic analysis. A third Δ_I incorporates the basic idea of endoscopy and transfer, weighted sums of orbital integrals, often referred to as κ-orbital integrals. The two others, Δ_1 and Δ_2, are cohomological in nature and are there to compensate the arbitrary elements that had to be used in the definition of Δ_{II}. It should be observed that Δ_I does not appear in [Sh] where it had to be replaced by an existence argument.

The definitions of Δ_1 and Δ_2 are quite elaborate, involving a number of general arguments and constructions that we have preferred to place in a separate Sect. 2 which also prepares for the product formula of Sect. 6.

Although we do not discuss the existence of the transfer, thus of f^H for a given f, the result of Sect. 5 is evidence that it will be available, and is in addition the source of the factors Δ_I and Δ_{II}. The limit formula of (5.5) shows that the dominant term near the identity of the combination of orbital integrals of f appearing in the transfer can be made equal to the dominant term of the stable orbital integral of an f^H. This is clearly a necessary condition for the existence of f^H, and is what guarantees that the choice of the correction factor Δ_I is correct. The factor Δ_2 does not affect the asymptotic behavior, and is dictated by experience with the real field.

The properties of the transfer factor that in addition to (5.5) in all probability characterize it are the Local Hypothesis, which relates the transfer factor on an arbitrary group to that on a quasi-split inner twisting, the Global Hypothesis, which is a product formula, and the transfer factor over archimedean fields, already introduced in [Sh]. We prove the first two here, in Sects. 4 and 6, reserving the proof that the transfer factor of this paper coincides over $\mathbf{R}$ with that of [Sh] for a later paper.

It is a pleasure to dedicate this paper to Friedrich Hirzebruch, for one of us first realized the significance of L-packets during a long stay in Bonn many years ago under the auspices of the SFB, when he was able to study Shimura's papers on automorphic forms. At the same time the other was beginning the study of character identities for real groups, and L-packets and character identities together led to endoscopy.

I. Preliminaries

(1.1) *An Example*

Suppose that F is a local field of characteristic zero and G is SL_2. Take H to be a one-dimensional torus split over the quadratic extension E of F and anisotropic over F. Then H is an endoscopic group for G. To $\gamma_H \in H(E)$

we attach the conjugacy class in $G(E)$ consisting of semisimple elements with eigenvalues $\gamma_H^{\pm 1}$. Assume that $\gamma_H \neq \pm 1$ lies in $H(F)$. Then the associated class meets $SL_2(F)$ in a stable conjugacy class of regular semisimple elements. Call γ_H *an image* of any γ in this stable class.

For $f \in C_c^\infty(G(F))$ form the integral $\Phi(\gamma, f)$ of f along the conjugacy class of regular semisiimple γ in $G(F)$ with respect to the G-invariant measure prescribed by a choice of invariant forms of highest degree over F on G and H. Then transfer between G and H requires a function Δ, a *transfer factor*, such that

$$f^H : \gamma_H \to \sum_\gamma \Delta(\gamma_H, \gamma)\Phi(\gamma, f)$$

extends smoothly to $\gamma_H = \pm 1$. Here $\sum_\gamma$ indicates summation over representatives for the regular semisimple conjugacy classes in $G(F)$; $\Delta(\gamma_H, \gamma)$ is to be zero unless γ_H is an image of γ, so that the sum contains at most two non-zero terms.

We modify slightly the prescription for Δ in [L-L]. For reasons of functoriality Δ will depend not on H alone but on a set $(H, \mathcal{H}, s, \zeta)$ of endoscopic data: $\mathcal{H} = \widehat{H} \rtimes W$ is the L-group of H, s lies in the conjugacy class of $\widehat{G} = PGL(2, \mathbf{C})$ determined by the elements of $GL_2(\mathbf{C})$ with eigenvalues $\pm z, z \in \mathbf{C}$, and ξ is an embedding of $\mathcal{H}$ in $\widehat{G} \rtimes W = {}^L G$ that carries $\widehat{H}$ into Cent $(s, \widehat{G})$. Here we may take for W the Weil group of E/F. Equivalent, or $\widehat{G}$-conjugate, data will yield the same factor Δ.

We first define a factor Δ_0 which depends in addition on the choice of an F-splitting of G. The quotients $\Delta_0(\gamma_H, \gamma)/\Delta_0(\overline{\gamma}_H, \overline{\gamma})$ will, however, be canonical. To prescribe Δ we fix some $(\overline{\gamma}_H, \overline{\gamma})$ with $\overline{\gamma}_H$ an image of $\overline{\gamma}$, specify $\Delta(\overline{\gamma}_H, \overline{\gamma})$ arbitrarily, and then set

$$\Delta(\gamma_H, \gamma) = \Delta(\overline{\gamma}_H, \overline{\gamma}) \frac{\Delta_0(\gamma_H, \gamma)}{\Delta_0(\overline{\gamma}_H, \overline{\gamma})} \tag{1.1.1}$$

if γ_H is an image of γ. Thus Δ is canonical, up to the constant $\Delta(\overline{\gamma}_H, \overline{\gamma})$.

The factor Δ_0 will be a product of several terms. Only the first depends on the choice of F-splitting. Here we will describe it for the standard splitting $(\mathbf{B}, \mathbf{T}, X)$: $\mathbf{B}$ is the upper triangular subgroup, $\mathbf{T}$ the diagonal subgroup and $X = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. For the general case and for the fact that the relative factor is independent of the choice of splitting we refer to (3.2).

Other choices are needed to define the terms in Δ_0: an *admissible embedding* of H in G, and *a-data* and *χ-data* for the image of H under that embedding. Let $\Gamma =$ Gal(E/F). An embedding of the torus H in G is admissible if it is dual to a composition $\widehat{H} \to \mathcal{T} \to \widehat{T}$, where $\mathcal{T}$ is some maximal torus in $\widehat{G}$ containing $s, \mathcal{T} \to \widehat{T}$ is the isomorphism attached to $(\mathcal{B}, \mathcal{T})$ and some pair (B, T) chosen so that T is defined over F and $\widehat{H} \to \widehat{T}$ is a Γ-isomorphism. See (1.2) and (1.3). The a-data for T consist of elements $a_\alpha, a_{-\alpha}$ of $E^\times$ such that $\overline{a}_\alpha = -a_\alpha = a_{-\alpha}$, where $\pm\alpha$ are the roots of T and the bar denotes conjugation in E.

The χ-data are characters $\chi_\alpha, \chi_{-\alpha}$ on $E^\times$ which extend the quadratic character on $F^\times$ attached to E and satisfy $\chi_{-\alpha} = \chi_\alpha^{-1}$.

Here then are the terms in Δ_0.

$$(I) \qquad \langle \lambda(T), s_T \rangle,$$

where $\lambda(T)$ is the class in $H^1(T) = H^1(\Gamma, T(E))$ of the cocycle

$$\sigma \to h \begin{bmatrix} 0 & a_\alpha \\ -a_\alpha^{-1} & 0 \end{bmatrix} \sigma(h^{-1})$$

for σ nontrivial in Γ. The element h is given by $h^{-1}Th = \mathbf{T}$ and $h^{-1}Bh = \mathbf{B}$, with B some Borel subgroup containing T. Then α is the root of T in B. The element s_T is the unique nontrivial Γ-invariant element in $\widehat{T}$; it is also the image of s under $\mathcal{T} \to \widehat{T}$. Finally, the pairing is the Tate-Nakayama pairing.

$$(II) \qquad \chi_\alpha \left(\frac{\alpha(\gamma_T) - 1}{a_\alpha} \right),$$

where γ_T is the image of γ_H under $H \to T$, and α is either root of T in G.

$$(III_1) \qquad \langle \mathrm{inv}(\gamma_H, \gamma), s_T \rangle^{-1},$$

where $\mathrm{inv}(\gamma_H, \gamma)$ is the class in $H^1(T)$ of the cocycle $\sigma \to g\sigma(g)^{-1}$ and g is given by $g^{-1}\gamma g = \gamma_T$.

$$(III_2) \qquad \langle \mathbf{a}, \gamma_T \rangle,$$

where $\mathbf{a}$ is an element of $H^1(W, \widehat{T})$ which, roughly speaking, measures the difference between the $\widehat{G}$-conjugacy class of the embedding $\xi : {}^LH \to {}^LG$ and the class of embeddings canonically attached to the χ-data for T [see (2.6)]. The pairing is the usual one between $H^1(W, \widehat{T})$ and $T(F)$ [B, Sect. 9].

$$(IV) \qquad |(\alpha(\gamma_T) - 1)(\alpha(\gamma_T)^{-1} - 1)|_F^{1/2}.$$

The image T of H in G may be replaced only by $g^{-1}Tg$, where $g \in \mathfrak{A}(T) = \{g \in G(E) : g\sigma(g^{-1}) \in T\}$. If $(T, \{a_\alpha\}, \{\chi_\alpha\})$ is replaced by a triple conjugate under $\mathfrak{A}(T)$ in the obvious sense then we see that only the terms (I) and (III_1) are affected, but then clearly the effects cancel. Thus it remains to consider T fixed and the a-data, χ-data changed. Only (I) and (II) involve a-data. If a_α is replaced by $a'_\alpha = a_\alpha b_\alpha$ where $b_\alpha \in F^\times$ then $\lambda(T)$ is multiplied by the class $\mathbf{b}$ of the cocycle $\sigma \to b_\alpha^{\alpha^\vee}$. Note that $b_\alpha^{\alpha^\vee}$ lies in $E^\times \otimes X_*(T) = T(E)$. To show that the product of the terms (I) and (II) is unaffected by replacing a_α with a'_α we have only to check that

$$\langle \mathbf{b}, s_T \rangle = \chi_\alpha(b_\alpha).$$

This is clear since **b** is trivial if and only if b lies in $\mathrm{Nm}E^\times$.

On the other hand, only the terms (II) and (III$_2$) depend on χ-data. Suppose that χ_α is replaced by $\chi'_\alpha = \chi_\alpha\zeta_\alpha$. Then ζ_α must be trivial on $F^\times$. We use the fact that the norm map $\delta \to \delta\overline{\delta}$ from $T(E)$ to $T(F)$ is surjective to write γ_T as $\delta_T\overline{\delta}_T$. Then $\alpha(\gamma_T) = \alpha(\delta_T)/\overline{\alpha(\delta_T)}$ and

$$\zeta_\alpha\left(\frac{\alpha(\delta_T)-1}{a_\alpha}\right) = \zeta_\alpha\left(\frac{\alpha(\delta_T)-\overline{\alpha(\delta_T)}}{a_\alpha}\right)\cdot\zeta_\alpha(\alpha(\delta_T))$$

or $\zeta_\alpha(\alpha(\delta_T))$ since $a_\alpha^{-1}(\alpha(\delta_T)-\overline{\alpha(\delta_T)})$ lies in $F^\times$. Thus we have to show that when χ_α is replaced by $\chi_\alpha\zeta_\alpha$ the class **a** appearing in the term (III$_2$) is multiplied by a class $\mathbf{a}(\zeta_\alpha)$ such that

$$\langle \mathbf{a}(\zeta_\alpha), \gamma_T\rangle = \zeta_\alpha(\alpha(\delta_T))^{-1}.$$

We will see later (3.5) that in fact **a** is multiplied by the class of a cocycle which is given on $E^\times$ by $\chi \to \zeta_\alpha(\chi)^{-\alpha}$. That the class pairs with γ_T in the desired way is the Base Change Identity in this simple case, which follows from the remarks in [B, Sect. 9].

Lemma [L-L]. *For $f \in C_c^\infty(G(F))$ the function*

$$f^H : \gamma_H \to \sum_\gamma \Delta(\gamma_H, \gamma)\Phi(\gamma, f)$$

extends smoothly to $H(F)$.

(1.2) *Notation*

Throughout F will be a local or a global field of characteristic zero; $\overline{F}$ will be an algebraic closure of F, and Γ and W_F the Galois group and Weil group of $\overline{F}/F$. Let G be a connected reductive group over F. Then G^* will denote a quasi-split inner form of G and ${}^LG$ the L-group of G. More precisely, we fix:

(i) *(G^*, ψ) with G^* quasi-split over F and $\psi : G \to G^*$ an inner twist, and*

(ii) *$(\widehat{G}, \varrho, \eta_G)$ with $\widehat{G}$ connected, reductive and defined over $\mathbf{C}$, ϱ an L-action of Γ on $\widehat{G}$, and $\eta_G : \Psi(G)^\vee \to \Psi(\widehat{G})$ a Γ-bijection.*

Here $\Psi(-)$ denotes canonical based root data (see [K$_2$]).

We have then for each pair (B, T) in G and $(\mathcal{B}, \mathcal{T})$ in $\widehat{G}$ a canonical isomorphism $\widehat{T} \to \mathcal{T}$, where by a *pair* we mean a Borel subgroup and a maximal torus contained in it.

As L-group data, that is, the data of (ii), for G^* we take $(\widehat{G}, \varrho, \eta_{G^*})$ where η_{G^*} is given by

$$\Psi(G^*)^\vee \xrightarrow{\psi} \Psi(G)^\vee \xrightarrow{\eta_G} \Psi(\widehat{G}).$$

Finally, ${}^L G$ will be the semidirect product $\widehat{G} \rtimes W_F$ with W_F acting by $W_F \to \Gamma \xrightarrow{\varrho} \mathrm{Aut}\, \widehat{G}$.

We shall specify endoscopic data in a way useful for extension to the twisted case [K-S]. First note that if $1 \to \widehat{G} \to \mathcal{G} \to W_F \to 1$ is a split extension then $\mathcal{G}$ is not necessarily an L-group. Nevertheless we attach to $\mathcal{G}$ an L-action $\varrho_{\mathcal{G}}$ on $\widehat{G}$ as follows. Let **spl** $= (\mathcal{B}, \mathcal{T}, \{X_\alpha\})$ be a splitting of $\widehat{G}$. We shall require that **spl** is preserved by the ϱ appearing in (ii), that is, it is a Γ-*splitting* (or F-splitting). If $c : W_F \to \mathcal{G}$ splits the extension then for $w \in W_F$ we multiply
Int $c(w)$ acting on $\widehat{G}$ by an element of Int $\widehat{G}$ to obtain $\varrho_{\mathcal{G}}(w)$ preserving **spl**. Then $\varrho_{\mathcal{G}}$ is an L-action which is independent of the choice for c. Clearly if $\varrho_{\mathcal{G}}$ coincides with ϱ for some choice of **spl** then it coincides for all choices.

Let $(H, \mathcal{H}, s, \xi)$ be endoscopic data for G. By this we will mean:

(i) *H is a quasi-split group over F.*

Its L-group data will be denoted $(\widehat{H}, \varrho_H, \eta_H)$, and ${}^L H = \widehat{H} \rtimes W_F$.

(ii) *$\mathcal{H}$ is a split extension of W_F by $\widehat{H}$ and $\varrho_{\mathcal{H}}$ coincides with ϱ_H.*

(iii) *s is a semisimple element of $\widehat{G}$.*

(iv) *$\xi : \mathcal{H} \to {}^L G$ is an L-homomorphism, that is, a homomorphism of extensions of W_F, such that*

(a) Int $s \circ \xi = a \otimes \xi$,

where a is a locally trivial 1-cocycle of W_F in the center $Z(\widehat{G})$ of $\widehat{G}$ if F is global, or a trivial 1-cocycle of W_F in $Z(\widehat{G})$ if F is local, and

(b) *$\xi|\widehat{H}$ is an isomorphism of $\widehat{H}$ with the connected component of the centralizer of s in $\widehat{G}$.*

Here $a \otimes \xi(h) = a(w(h))\xi(h), h \in \mathcal{H}$, with $w(h)$ denoting the image of h under $\mathcal{H} \to W_F$.

Data $(H', \mathcal{H}', s', \xi')$ are equivalent to $(H, \mathcal{H}, s, \xi)$ *if there exist an F-isomorphism $\alpha : H \to H'$, an L-isomorphism $\beta : \mathcal{H}' \to \mathcal{H}$ and an element g of $\widehat{G}$ such that:*

(i) $\Psi(H) \xrightarrow{\alpha} \Psi(H')$ *and* $\Psi(\widehat{H}') \xrightarrow{\beta} \Psi(\widehat{H})$ *are dual,*

(ii) Int $g \circ \xi \circ \beta = \xi'$ *and:*

(iii) gsg^{-1} *lies in* $Z(\widehat{G})Z(\xi')^0 s'$, *where* $Z(\xi')$ *is the centralizer in* G *of the image of* $\mathcal{H}'$ *under* ξ'.

Up to equivalence and the choice of ξ, which amounts to the choice of an embedding of ${}^L H$ in ${}^L G$ in the case $\mathcal{H}$ is an L-group, these are the endoscopic data of [L2, p. 20]. Note that in the definitions of [L2] the group $\mathcal{H}$ generated by ${}^L H^0$ and the elements $n(w), w \in W_F$, is a split extension of W_F by ${}^L H^0$ (see L1, Lemma 4]).

If $\mathcal{H}$ is an L-group then we may assume that $\mathcal{H} = {}^L H$. *This will be our assumption until* (4.4) as it greatly simplifies notation. The minor modification needed for the general case, a passage to certain central extensions of H, will be dealt with in that section.

(1.3) *Point Correspondences*

The isomorphism $\widehat{T}_G \to \mathcal{T}_G$ attached to pairs (B_G, T_G) in G and $(\mathcal{B}_G, \mathcal{T}_G)$ in $\widehat{G}$ transports the coroots of T_G in G to the roots of $\mathcal{T}_G$ in $\widehat{G}$, the B_G-simple coroots to the $\mathcal{B}_G$-simple roots and the Weyl group of T, with contragredient action, to the Weyl group of $\mathcal{T}_G$. If $(\mathcal{B}_H, \mathcal{T}_H)$ is a pair in $\widehat{H}$ then there is an x in $\widehat{G}$ such that $\mathrm{Int} x \circ \xi$ maps $\mathcal{T}_H$ to $\mathcal{T}_G$ and $\mathcal{B}_H$ into $\mathcal{B}_G$. Finally, if (B_H, T_H) is a pair in H then we have an isomorphism $\widehat{T}_H \to \widehat{T}_G$ defined by the composition $\widehat{T}_H \to \mathcal{T}_H \longrightarrow \mathcal{T}_G \longrightarrow \widehat{T}_G$ and thus also an isomorphism $T_H \longrightarrow T_G$. These isomorphisms transport the coroots of T_H in H into a subsystem of the coroots of T_G in G, the Weyl group Ω_H of T_H into a subgroup of the Weyl group Ω_G of T_G, and the roots of T_H into a subset of the roots of T_G. The map

$$T_H/\Omega_H \longrightarrow T_G/\Omega_G$$

of orbits of Ω_H in T_H onto orbits of Ω_G in T_G is independent of all choices. Since these orbits classify the conjugacy classes of semisimple elements in $H(\overline{F})$ and $G(\overline{F})$, and the choice of tori is of no consequence, we have a canonical map

$$\mathcal{A}_{H/G} : C\ell_{\mathrm{ss}}(H(\overline{F})) \longrightarrow C\ell_{\mathrm{ss}}(G(\overline{F})).$$

We call semisimple $\gamma_H \in H(\overline{F})$ *G-regular* if the image of its conjugacy class under $\mathcal{A}_{H/G}$ consists of regular semisimple elements, and *strongly G-regular* if the image consists of strongly regular elements, that is, elements whose centralizer is a torus. A strongly G-regular element is strongly regular.

The group $\Gamma = \mathrm{Gal}(\overline{F}/F)$ acts on conjugacy classes.

Lemma 1.3.A. *$\mathcal{A}_{H/G}$ is a Γ-map.*

Proof. $\mathcal{A}_{H/G} = \mathcal{A}_{G^*/G} \cdot \mathcal{A}_{H/G^*}$ and $\mathcal{A}_{G^*/G}$ is the map induced by ψ. Since ψ is an inner twist $\mathcal{A}_{G^*/G}$ is a Γ-bijection. Thus we may assume that G is quasi-split over F. Then if T_H is defined over F Steinberg's Theorem [K1] allows us to choose (B_G, T_G) with both T_G and $T_H \longrightarrow T_G$ defined over F. The lemma follows.

Suppose that T_H is defined over F. If (B_{G^*}, T_{G^*}) is chosen so that T_{G^*}, and $T_H \longrightarrow T_{G^*}$ are defined over F, as in the proof of the lemma, then we call $T_H \longrightarrow T_{G^*}$ an *admissible embedding* of T_H in G^*. It is uniquely determined up to $\mathfrak{A}(T_{G^*})$-conjugacy, that is, up to composition with $\mathrm{Int}\ g^{-1}$, where g lies in

$$\mathfrak{A}(T_{G^*}) = \{g \in G^*(\overline{F}) : g\sigma(g^{-1}) \in T_{G^*}(\overline{F}), \sigma \in \Gamma\}.$$

Note that we may take g in G^*_{sc}, the simply-connected cover of the derived group of G^*.

For strongly regular elements in $G(F)$ or $H(F)$ stable conjugacy is the same as conjugacy under $G(\overline{F})$ or $H(\overline{F})$, and we may apply Lemma 1.3.A directly to define a correspondence of points. Thus, if $\gamma_H \in H(F)$ is strongly G-regular then its stable conjugacy class consists of the F-rational points in its conjugacy class in $H(\overline{F})$.

The image of this class in $H(\overline{F})$ under $\mathcal{A}_{H/G}$ is a conjugacy class of strongly regular elements in $G(\overline{F})$. The class is defined over F and so its F-rational points are either nonexistent or form a single stable conjugacy class of strongly regular elements in $G(F)$. We call strongly G-regular $\gamma_H \in H(F)$ *an image of* $\gamma_G \in G(F)$ if γ_G lies in the image under $\mathcal{A}_{H/G}$ of the conjugacy class of γ_H in $H(\overline{F})$. The twisted analogue of *image* is *norm* [K-S] which explains why we have labelled γ_H, and not γ_G, as the image.

For arbitrary G-regular semisimple γ_H in $H(F)$ we set $T_H = \mathrm{Cent}(\gamma_H, H)^0$ and choose an admissible embedding $T_H \longrightarrow T_{G^*}$ of T_H in G^*. If γ_G is regular semisimple in $G(F)$ and $T_G = \mathrm{Cent}(\gamma, G)^0$ then we say that γ_H *is an image of* γ_G if there exists $x \in G^*$ such that $\mathrm{Int}\; x \circ \psi$ maps γ_G to the image γ_{G^*} of γ_H under $T_H \longrightarrow T_{G^*}$ *and* T_G to T_{G^*}. The correspondence (γ_H, γ_G) is independent of the choice of admissible embedding $T_H \longrightarrow T_{G^*}$ and extends that for the strongly regular elements. Further we have:

(i) a G-regular semisimple element of $H(F)$ is either the image of no element or the image of exactly one stable conjugacy class of regular semisimple elements in $G(F)$, and

(ii) the images of a regular semisimple element in $G(F)$ form a union, possibly empty, of stable conjugacy classes of G-regular semisimple elements in $H(F)$. If F is local then the union is finite.

(1.4) *Transfer Factors*

We assume here that F is local, leaving remarks on the global case for Sect. 6.

To normalize measures on conjugacy classes we fix invariant forms of highest degree: ω_G on G, ω_H on H and $\omega_{\mathbf{T}}$ on some maximal torus $\mathbf{T}$ in G. Then if T is any maximal torus over F in either G or H we transport $\omega_{\mathbf{T}}$ to an invariant form ω_T of highest degree on T, using an inner automorphism of G if T lies in G and an isomorphism provided by the choice of pairs otherwise. In either case ω_T depends only on $\omega_{\mathbf{T}}$.

To an invariant form ω of highest degree on G, H or T we attach a Haar measure as follows. There is $\lambda_\sigma \in \overline{F}^\times$ such that $\sigma(\omega) = \lambda_\sigma \omega, \sigma \in \Gamma$. Hilbert's Theorem 90 allows us to write λ_σ as $\mu\sigma(\mu^{-1})$, where $\mu \in \overline{F}^\times$. Then $\mu\omega$ is defined over F and the Haar measure $|\mu\omega|$ is well-defined. To obtain a measure independent of the choice of μ we replace this by $|\mu|^{-1}|\mu\omega|$ which will be denoted simply as $|\omega|$.

It is simplest, and sufficient, to specify transfer factors on *strongly* regular elements. If $f \in C_c^\infty(G(F))$ then for the integral of f along the conjugacy class of any regular semisimple element γ in $G(F)$ we take

$$\Phi(\gamma, f) = \int\limits_{T(F)\backslash G(F)} f(g^{-1}\gamma g) \frac{|\omega_G|}{|\omega_T|},$$

where $T = \text{Cent}(\gamma, G)^0$. For the integral of $f^H \in C_c^\infty(H(F))$ along the stable conjugacy class of strongly G-regular γ_H in $H(F)$ we take

$$\Phi^{\text{st}}(\gamma_H, f^H) = \Sigma\Phi(\gamma'_H, f^H),$$

where the summation is over representatives γ'_H for the conjugacy classes in the stable conjugacy class of γ_H. If γ_H is not strongly regular this must be modified [see (4.3)].

Suppose Δ is a function on

$$\left\{ \begin{array}{l} \text{strongly } G\text{-regular} \\ \text{elements in } H(F) \end{array} \right\} \times \left\{ \begin{array}{l} \text{strongly regular} \\ \text{elements in } G(F) \end{array} \right\}$$

such that

(i) $\Delta(\gamma_H, \gamma)$ depends only on the conjugacy class of γ in $G(F)$ and the stable conjugacy class of γ_H in $H(F)$, and

(ii) $\Delta(\gamma_H, \gamma) = 0$ unless γ_H is an image of γ.

Then we say that $f \in C_c^\infty(G(F))$ and $f^H \in C_c^\infty(H(F))$ have Δ-*matching orbital integrals* if

$$\Phi^{\text{st}}(\gamma_H, f^H) = \sum_\gamma \Delta(\gamma_H, \gamma)\Phi(\gamma, f)$$

for all strongly G-regular elements γ_H in $H(F)$. The summation is over representatives γ for the conjugacy classes of strongly regular elements in $G(F)$; only a finite number of terms in the sum are nonzero.

We call Δ a *transfer factor* if for each $f \in C_c^\infty(G(F))$ there exists $f^H \in C_c^\infty(H(F))$ with Δ-matching orbital integrals. It is best to demand that $\Delta(\gamma_H, \gamma)$ be nonzero if γ_H is an image of γ, and sometimes preferable to work with functions in the Schwartz space [Sh].

In Sect. 3 we will define a function Δ. If G is quasi-split over F then the procedure is exactly that for SL(2). In general, however, the term inv(γ_H, γ) appearing in (III$_1$) of (1.1) is no longer well defined as the torus T will be taken in G^* rather than G. Since only quotients really matter we define instead a relative invariant following [K-S], and obtain a canonical relative factor $\Delta(\gamma_H, \gamma; \overline{\gamma}_H, \overline{\gamma})$ in place of the quotient $\Delta_0(\gamma_H, \gamma)/\Delta_0(\overline{\gamma}_H, \overline{\gamma})$ in (1.1.1).

The next section describes, in a general setting, two constructions needed for Sect. 3.

2. Key Lemmas

(2.1) *General Remarks*

Suppose that k is a field of characteristic zero with algebraic closure $\overline{k}$, and that G is a connected reductive algebraic group defined and quasi-split over k. Let $(\mathbf{B}, \mathbf{T}, \{X_\alpha\})$ be a k-splitting of G and $\boldsymbol{\Gamma}$ be a group acting on G by automorphisms which preserve $(\mathbf{B}, \mathbf{T}, \{X_\alpha\})$. Then $\boldsymbol{\Gamma}$ acts on the Weyl group $\boldsymbol{\Omega} = \Omega(G, \mathbf{T})$ of $\mathbf{T}$ in G. For $\theta = \omega \rtimes \varrho \in \Omega \rtimes \boldsymbol{\Gamma}$ we define $n(\theta) = n(\omega) \rtimes \varrho \in G(\overline{k}) \rtimes \boldsymbol{\Gamma}$, where $n(\omega) \in$ Norm $(\mathbf{T}(\overline{k}), G(\overline{k}))$ is given by the following well-known construction [Spr].

Let α be a simple root of $\mathbf{T}$ in $\mathbf{B}$. Then by definition $\varrho X_\alpha = X_{\varrho\alpha}, \varrho \in \Gamma$. Let H_α be the coroot for α regarded as an element of Lie $\mathbf{T}$. Fix the root vector $X_{-\alpha}$ for $-\alpha$ by requiring that $[X_\alpha, X_{-\alpha}] = H_\alpha$. Then $\varrho X_{-\alpha} = X_{-\varrho\alpha}, \varrho \in \Gamma$. Set

$$n(\alpha) = \exp X_\alpha \exp(-X_{-\alpha}) \exp X_\alpha,$$

so that $n(\alpha)$ is the image of $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ under the homomorphism $\mathrm{SL}(2) \to G$ attached to the Lie triple $\{X_\alpha, H_\alpha, X_{-\alpha}\}$. Let $\omega \in \boldsymbol{\Omega}, \omega \neq 1$, be written in reduced form $\omega(\alpha_1) \ldots \omega(\alpha_r)$. Then we may set $n(\omega) = n(\alpha_1) \ldots n(\alpha_r)$ since this latter product is independent of the choice of reduced expression for ω [Spr]. If $\omega = 1$ set $n(\omega) = 1$. Then $\mathrm{Int} n(\omega)$ acts on $\mathbf{T}$ as ω and

$$n(\varrho(\omega)) = \varrho(n(\omega)), \qquad \varrho \in \boldsymbol{\Gamma}.$$

Thus if $\theta = \omega \rtimes \varrho$ then $n(\theta) = n(\omega) \rtimes \varrho$ acts on T as θ and

$$n(\theta_1) n n(\theta_2) = t(\theta_1, \theta_2) n(\theta_1 \theta_2), \quad \theta_i \in \boldsymbol{\Omega} \rtimes \boldsymbol{\Gamma},$$

where $t(\theta_1, \theta_2)$ is a 2-cocycle of $\boldsymbol{\Omega} \rtimes \boldsymbol{\Gamma}$ in $\mathbf{T}(\overline{k})$.

Lemma 2.1.A.

$$t(\theta_1, \theta_2) = \prod_{\substack{\alpha > 0 \\ \theta_1^{-1}\alpha < 0 \\ \theta_2^{-1}\theta_1^{-1}\alpha > 0}} (-1)^{\alpha^\vee}.$$

Here $\alpha > 0$ means α is a root of $\mathbf{T}$ in $\mathbf{B}$, and $\alpha^\vee$ is the coroot for α as element of $X_*(\mathbf{T})$. Then $(-1)^{\alpha^\vee} \in k^\times \otimes X_*(\mathbf{T}) \subseteq \mathbf{T}(\overline{k})$.

Proof. We will verify in the next lemma that the right side is, like the left, a 2-cocycle. It is therefore enough to show that the equality is valid when (i) θ_1 or θ_2 lies in Γ and when (ii) θ_1 and θ_2 lie in Ω and further $\theta_1 = \omega(\alpha)$ where α is simple. Case (i) is clear. For case (ii), let $\omega = \omega(\alpha_1) \ldots \omega(\alpha_r)$ be a reduced expression for θ_2. Then $R_\omega = \{\beta > 0, \omega\beta < 0\}$ is the set of positive roots separating the positive Weyl chamber W_+ from $\omega^{-1} W_+$ and contains r elements. There are two possibilities. Either $\omega^{-1}\alpha > 0, \omega(\alpha)\omega(\alpha_1) \ldots \omega(\alpha_r)$ is a reduced expression for $\omega(\alpha)\omega$ and $R_{\omega(\alpha)\omega} = R_\omega \cup \{\omega^{-1}\alpha\}$, or $\omega^{-1}\alpha < 0$ and there is a reduced expression for ω with $\omega(\alpha_1) = \omega(\alpha)$. In the first case both sides of the putative equality are 1; in the second both are $(-1)^{\alpha^\vee}$. The lemma is thus proved.

Let X be a free finitely generated $\mathbf{Z}$-module and Σ be a group which acts on X and contains an element ϵ sending λ to $-\lambda, \lambda \in X$. Then with trivial action of $k^\times, \Sigma$ acts on $k^\times \otimes X$. Let $\mathcal{R}$ be a finite Σ-stable subset of

X and p be a *gauge* on $\mathcal{R}$, by which we mean that $p : \mathcal{R} \to \{\pm 1\}$ and $p\{-\lambda) = -p(\lambda), \lambda \in \mathcal{R}$. We abbreviate a product over those $\lambda \in \mathcal{R}$ for which

$$p(\lambda) = 1, \quad p(\sigma^{-1}\lambda) = -1, \quad p(\tau^{-1}\sigma^{-1}\lambda) = 1$$

by $\prod\limits_{1,\sigma,\tau}^{p}$.

Lemma 2.1.B. $$t_p(\sigma,\tau) = \prod_{1,\sigma,\tau}^{p} (-1)^{\lambda}, \ \sigma, \tau \in \Sigma,$$

is a 2-cocycle of Σ with values in $k^\times \otimes X$.

Proof. The coboundary of this 2-cochain is the product over all λ with $p(\lambda) = 1$ of $(-1)^{N\lambda}$, where N is the number of ordered triples in

$$\{(\sigma^{-1}\lambda, \tau^{-1}\sigma^{-1}\lambda, \varrho^{-1}\tau^{-1}\sigma^{-1}\lambda),(\lambda, \tau^{-1}\sigma^{-1}\lambda, \varrho^{-1}\tau^{-1}\sigma^{-1}\lambda),$$
$$(\lambda, \sigma^{-1}\lambda, \varrho^{-1}\tau^{-1}\sigma^{-1}\lambda),(\lambda, \sigma^{-1}\lambda, \tau^{-1}\sigma^{-1}\lambda)\}$$

on which p takes alternating signs. Thus if $p(\varrho^{-1}\tau^{-1}\sigma^{-1}\lambda) = -1$ there can be a contribution only from the first and last triples and then only if $p(\sigma^{-1}\lambda) = -1, p(\tau^{-1}\sigma^{-1}\lambda) = 1$ when both triples contribute, so that $N = 2$. If $p(\varrho^{-1}\tau^{-1}\sigma^{-1}\lambda) = 1$ and $p(\sigma^{-1}\lambda) = 1$ then either the first and second triples contribute or none does, so that N is 2 or 0, but if $p(\varrho^{-1}\tau^{-1}\sigma^{-1}\lambda) = 1$ and $p(\sigma^{-1}\lambda) = -1$ then the third triple contributes and exactly one of the second and fourth, so that N is 2. The lemma is proved.

Lemma 2.1.C. *If q is also a gauge on $\mathcal{R}$, then t_q is cohomologous to t_p.*

Proof. We may assume that Σ is transitive on $\mathcal{R}$ and that X is free on $\{\lambda : p(\lambda) = 1\}$. Then by Shapiro's Lemma we reduce to the case $\mathcal{R} = \{\pm\lambda\}$ where the assertion is clear.

In the application of (2.3) and (2.6) we will have $\epsilon^2 = 1$ and Σ will be the product of $\{1, \epsilon\}$ and a subgroup Γ. Then if Σ acts transitively on $\mathcal{R}$ either $\mathcal{R}$ consists of exactly two Γ-orbits $\mathcal{O}$ and $-\mathcal{O}$ or Γ also acts transitively on $\mathcal{R}$, in which case $\mathcal{R}$ is a Γ-orbit $\mathcal{O}$, where $\mathcal{O} = -\mathcal{O}$. In the former case $\mathcal{O}$ is called *asymmetric* and in the latter *symmetric*.

Lemma 2.1.D. *Suppose that $\mathcal{R} = \cup \pm \mathcal{O}$, where $\mathcal{O}$ is asymmetric. Then the restriction of t_p is trivial.*

Proof. We may assume X is free on $\mathcal{O}$. Then again by Shapiro's Lemma we reduce to the case $\mathcal{R} = \{\pm\lambda\}$ where the assertion is clear.

(2.2) *a-Data*

First we consider splitting t_p in $\overline{k}^\times \otimes X$, given an extension of the action of Σ on k to $\overline{k}$ with ϵ acting trivially. A collection $\{a_\lambda : \lambda \in \mathcal{R}\}$ is a set of *a-data* for the action of Γ in $\mathcal{R}$ if:

(i) $a_\lambda \in \overline{k}^\times$ *and* $a_{\sigma\lambda} = \sigma(a_\lambda), \sigma \in \Gamma, \lambda \in \mathcal{R}$, *and*

(ii) $a_{-\lambda} = -a_\lambda, \lambda \in \mathcal{R}$.

Of course, a-data need not exist. If they do then we can split t_p on Γ in the following way.

For a product over those $\lambda \in \mathcal{R}$ such that $p(\sigma^{-1}\lambda) = 1$ and $p(\tau^{-1}\lambda) = -1$ we write $\prod\limits_{\sigma,\tau}^{p}$. Set

$$u_p(\sigma) = \prod_{1,\sigma}^{p} a_\lambda^\lambda, \quad \sigma \in \Gamma.$$

Lemma 2.2.A. $\quad t_p(\sigma,\tau) = \partial u_p(\sigma,\tau), \quad \sigma,\tau \in \Gamma.$

Proof. $\partial u_p(\sigma,\tau) = u_p(\sigma)\sigma(u_p(\tau))u_p(\sigma\tau)^{-1} = \prod\limits_{1,\sigma}^{p} a_\lambda^\lambda \prod\limits_{\sigma,\tau}^{p} a_\lambda^\lambda \prod\limits_{1,\sigma\tau}^{p} a_\lambda^{-\lambda}$.

Fix $\lambda \in \mathcal{R}$ with $p(\lambda) = 1$. Then the contributions of the terms with exponents $\pm\lambda$ are as follows.

(i) $p(\sigma^{-1}\lambda) = 1, \quad p(\tau^{-1}\sigma^{-1}\lambda) = 1 : 1$

(ii) $p(\sigma^{-1}\lambda) = 1, \quad p(\tau^{-1}\sigma^{-1}\lambda) = -1 : a_\lambda^\lambda \cdot a_\lambda^{-\lambda} = 1$

(iii) $p(\sigma^{-1}\lambda) = -1, p(\tau^{-1}\sigma^{-1}\lambda) = 1 : a_\lambda^\lambda \cdot a_{-\lambda}^{-\lambda} = a_\lambda^\lambda(-a_\lambda)^{-\lambda} = (-1)^\lambda$

(iv) $p(\sigma^{-1}\lambda) = -1, p(\tau^{-1}\sigma^{-1}\lambda) = -1 : 1$.

Thus $\partial u_p(\sigma,\tau) = \prod\limits_{1,\sigma,\tau}^{p} (-1)^\lambda = t_p(\sigma,\tau)$.

As a corollary of this lemma, or by simply modifying the proof, we have:

Lemma 2.2.B. *Suppose that* $\{b_\lambda : \lambda \in \mathcal{R}\}$ *satisfies* $b_\lambda \in \overline{k}^\times$,

$$b_{\sigma\lambda} = \sigma(b_\lambda), \quad \sigma \in \Gamma, \ \textit{and}\ b_{-\lambda} = b_\lambda.$$

Then

$$v_p(\sigma) = \prod_{1,\sigma}^{p} b_\lambda^\lambda, \quad \sigma \in \Gamma,$$

is a 1-cocycle of Γ *in* $\overline{k}^\times \otimes X$.

By the usual argument with Shapiro's Lemma we have:

Lemma 2.2.C. (a) *The class of* v_p *is independent of* p, *and*

(b) *if* $\mathcal{R} = \pm\mathcal{O}$ *where* $\mathcal{O}$ *is an asymmetric* Γ*-orbit then* v_p *is cohomologically trivial.*

Proof. (a) We may assume that $\mathcal{R} = \{\pm\lambda\}$. Then the only gauges are $\pm p$, where $p(\lambda) = 1$. Since $b_{-\lambda}^{-\lambda} = b_\lambda^\lambda b_\lambda^{-2\lambda}$ we have that $v_{-p} = v_p w$, where $w(\sigma) = b_\lambda^{-2\lambda}$ if $\sigma\lambda = -\lambda$ and $w(\sigma) = 1$ otherwise. But $w(\sigma) = x\sigma(x)^{-1}$ where $x = b_\lambda^{-\lambda}$.

(b) Again assume $\mathcal{R} = \{\pm\lambda\}$. Then $v_p = 1$.

(2.3) *An Application*

Our first task is to define an invariant for a pair $(T, \{a_\alpha\})$, where T is a maximal torus defined over k in a connected reductive group G defined and quasi-split over k, and $\{a_\alpha\}$ is a set of a-data for T, i.e. for the action of $\Gamma = \mathrm{Gal}(\overline{k}/k)$ on the set $R(G,T)$ of roots of T in G. Note that in this setting a-data are readily verified to exist.

We fix a k-splitting $(\mathbf{B}, \mathbf{T}, \{X_\alpha\})$ of G. There will be no harm in assuming G semisimple and simply-connected and we do so to conserve notation. Denote by $\boldsymbol{\sigma}$ the action of $\sigma \in \Gamma$ on $\mathbf{T}$, and set $\boldsymbol{\Gamma} = \{\boldsymbol{\sigma} : \sigma \in \Gamma\}$. Again $\boldsymbol{\Omega}$ will be the Weyl group $\Omega(G, \mathbf{T})$.

Choose a Borel subgroup B of G containing T and $h \in G$ such that $(B,T)^h = (\mathbf{B}, \mathbf{T})$. Denote by σ_T both the action of $\sigma \in \Gamma$ on T and its transport to $\mathbf{T}$ by $\mathrm{Int}\ h^{-1}$. Set

$$\Gamma_T = \{\sigma_T : \sigma \in \Gamma\} \subset \boldsymbol{\Omega} \rtimes \boldsymbol{\Gamma}.$$

If $\omega_T(\sigma)$ is the element of $\boldsymbol{\Omega}$ defined by $\mathrm{Int}(h^{-1}\sigma(h))$ then $\sigma_T = \omega_T(\sigma) \rtimes \boldsymbol{\sigma}$.

The a-data for T serve also for the action on Γ on $R^\vee(G,T)$ and, after transport, for the action of Γ_T on $\mathcal{R} = R^\vee(G, \mathbf{T})$; $\{a_\alpha^{-1}\}$ is also a set of a-data. By Lemmas 2.1.A and 2.2.A

$$n(\sigma_T) = n(\omega_T(\sigma)) \rtimes \boldsymbol{\sigma}, \quad \sigma \in \Gamma,$$

satisfies

$$n(\sigma_T)\sigma(n(\tau_T))n((\sigma\tau)_T)^{-1} = \partial x(\sigma_T, \tau_T)^{-1}$$

where

$$x(\sigma_T) = \prod_{1,\sigma}^{p} a_\alpha^{\alpha^\vee}$$

and $p(\alpha) = 1$ if and only if α is a root of $\mathbf{T}$ in $\mathbf{B}$. Then

$$\sigma_T \to x(\sigma_T) n(\omega_T(\sigma)) \rtimes \sigma$$

is a homomorphism of Γ_T in $\mathrm{Norm}(\mathbf{T}(k), G(k)) \rtimes \boldsymbol{\Gamma}$. Otherwise stated,

$$\sigma_T \to m(\sigma_T) = x(\sigma_T) n(\omega_T(\sigma))$$

is a 1-cocycle of Γ_T in $\mathrm{Norm}(\mathbf{T}(\overline{k}), G(\overline{k}))$.

Now $hm(\sigma_T)\sigma(h^{-1}) = h(m(\sigma_T)\sigma(h^{-1})h)h^{-1}$ lies in $T(\overline{k})$ and is evidently a 1-cocycle of Γ in $T(\overline{k})$. Since h is unique up to left multiplication by an element of $T(\overline{k})$ this cocycle yields, for given k-splitting, a-data and $B \supset T$, a well-defined element $\lambda(T) = \lambda_{\{a_\alpha\}}(T)$ of $H^1(\Gamma, T(\overline{k})) = H^1(T)$. We note some of its properties.

(2.3.1) The splitting $(\mathbf{B}, \mathbf{T}, \{\mathbf{X}_\alpha\})$ may be replaced only by $(\mathbf{B}^g, \mathbf{T}^g, \{\mathbf{X}_\alpha^g\})$, where $g \in G(\overline{k})$ is such that $g\sigma(g)^{-1}$ lies in the center Z of $G, \sigma \in \Gamma$. Then $m(\sigma_T)$ is replaced by $g^{-1}m(\sigma_T)g$ and h by hg. Thus the cocycle defining $\lambda(T)$ is replaced by

$$hg(g^{-1}m_T(\sigma)g)\sigma(hg)^{-1} = g\sigma(g)^{-1}hm_T(\sigma)\sigma(h)^{-1}.$$

Then $\lambda(T)$ is multiplied by the class **g** in $H^1(T)$ represented by $\sigma \to g\sigma(g)^{-1}$.

(2.3.2) Suppose that the a-data $\{a_\alpha\}$ are replaced by $\{a'_\alpha\}$. Then $a'_\alpha = b_\alpha a_\alpha$, where $b_{\sigma\alpha} = \sigma(b_\alpha)$ for all σ in the group Σ generated by Γ and ϵ, and $hm(\sigma_T)\sigma(h^{-1})$ is replaced by

$$hv_p(\sigma_T)h^{-1}hm(\sigma_T)\sigma(h^{-1}),$$

where v_p is the 1-cocycle $\sigma_T \to \prod_{1,\sigma}^{p} b_\alpha^{\alpha^\vee}$ of Γ_T in $\mathbf{T}(\overline{k})$ (cf. Lemma 2.2.B). Note that $hv_p(\sigma_T)h^{-1}$ is the cocycle $b_q : \sigma \prod_{1,\sigma}^{q} b_\alpha^{\alpha^\vee}$ of Γ in $T(\overline{k})$, q denoting the transport of the gauge p to $R^\vee(G,T)$ by $\mathrm{Int}\ h$, and that the class **b** of b_q in $H^1(T)$ is independent of the gauge q (Lemma 2.2.C).

(2.3.3) Next we show that $\lambda(T)$ is independent of B. Suppose that B is replaced by $vBv^{-1}, v \in \mathrm{Norm}(T,G)$. Set $u = h^{-1}vh \in \mathrm{Norm}(\mathbf{T}, G)$, and suppose that μ is the element of $\boldsymbol{\Omega}$ defined by u. We now have to transport σ on T to $\mathbf{T}$ by $\mathrm{Int}(h^{-1}v^{-1})$. Suppose we obtain $\sigma'_T = \omega'_T(\sigma) \rtimes \boldsymbol{\sigma}$. Then since $h^{-1}v^{-1}\sigma(vh) = h^{-1}v^{-1}h \cdot h^{-1}\sigma(h) \cdot \sigma(h^{-1}vh) = u^{-1}h^{-1}\sigma(h)\sigma(u)$ we have $\omega'_T(\sigma) = \mu^{-1}\omega_T(\sigma)\sigma(\mu)$. Let $\Gamma'_T = \{\sigma'_T : \alpha \in \Gamma\}$. The a-data $\{a'_\alpha\}$ obtained by transport for Γ'_T satisfy $a'_\alpha = a_{\mu\alpha}, \alpha \in R(G, \mathbf{T})$. We then define $m(\sigma'_T)$ in the same way as $m(\sigma_T)$ and consider

$$vhm(\sigma'_T)\sigma(h^{-1}v^{-1}) = hum(\sigma'_T)\sigma(u^{-1})\sigma(h^{-1}).$$

But now $um(\sigma'_T)\sigma(u)^{-1} = b(\sigma_T)m(\sigma_T)$, where $\sigma_T \to b(\sigma_T)$ is a 1-cocycle of Γ_T with values in $\mathbf{T}$. It remains to show:

Lemma 2.3.A. *The cocycle b is trivial.*

Proof. We have $m(\sigma_T) = x(\sigma_T)n(\omega_T(\sigma))$ and $m(\sigma'_T) = x(\sigma'_T)n(\omega'_T(\sigma))$; $x(\sigma_T)$ is defined in terms of $\{a_\alpha\}$ and $x(\sigma'_T)$ in terms of $\{a'_\alpha\}$. Then

$$m(\sigma'_T) = x(\sigma'_T)n(\mu^{-1}\omega_T(\sigma)\boldsymbol{\sigma}(\mu))$$

which equals

$$x(\sigma'_T)t_p(\mu^{-1}\omega_T(\sigma), \boldsymbol{\sigma}(\mu))t_p(\mu^{-1}, \omega_T(\sigma))n(\mu^{-1})n(\omega_T(\sigma))n(\boldsymbol{\sigma}(\mu))$$

or

$$x(\sigma'_T)\mu^{-1}(x(\sigma_T)^{-1})t_p(\mu^{-1}\omega_T(\sigma), \boldsymbol{\sigma}(\mu))t_p(\mu^{-1}, \omega_T(\sigma))n(\mu^{-1})m(\sigma_T)n(\boldsymbol{\sigma}(\mu)).$$

Now $n(\mu) = \lambda u$, where $\lambda \in \mathbf{T}$, and then

$$n(\boldsymbol{\sigma}(\mu)) = \sigma(n(\mu)) = \sigma(\lambda)\sigma(u).$$

Also we have $n(\mu^{-1}) = n(\mu)^{-1} t_p(\mu, \mu^{-1})$. Thus $um(\sigma'_T)\sigma(u)^{-1}$ is equal to

$$\mu(x(\sigma'_T))x(\sigma_T)^{-1} t_p(\mu, \mu^{-1})\mu[t_p(\mu^{-1}\omega_T(\sigma), \boldsymbol{\sigma}(\mu))t_0(\mu^{-1}, \omega_T(\sigma))]\lambda^{-1}\sigma_T(\lambda)m(\sigma_T),$$

and $b(\sigma_T)$ is equivalent to

$$\mu(x(\sigma'_T))x(\sigma_T)^{-1} t_p(\mu, \mu^{-1})\mu[t_p(\mu^{-1}\omega_T(\sigma), \sigma(\mu))t_p(\mu^{-1}, \omega_T(\sigma))].$$

We now omit the subscript T in notation. That $b(\sigma)$ is trivial follows from:

Lemma 2.3.B. (a) *Let* $\delta = \prod_{1,\mu}^{p} a_\alpha^{-\alpha^\vee}$. *Then* $\mu(x(\sigma'))x(\sigma)^{-1}$ *is equal to*

$$\delta\sigma(\delta^{-1}) \prod_{\substack{p(\alpha)=p(\mu^{-1}\alpha)=p(\mu^{-1}\sigma^{-1}\alpha)=1 \\ p(\sigma^{-1}\alpha)=-1}} (-1)^{\alpha^\vee} \prod_{\substack{p(\alpha)=p(\sigma^{-1}\alpha)=p(\mu^{-1}\sigma^{-1}\alpha)=1 \\ p(\mu^{-1}\alpha)=-1}} (-1)^{\alpha^\vee}$$

(b) $t_p(\mu, \mu^{-1})\mu[t_p(\mu^{-1}\omega(\sigma), \boldsymbol{\sigma}(\mu))t_p(\mu^{-1}, \omega(\sigma))]$ *is equal to*

$$\prod_{\substack{p(\alpha)=p(\mu^{-1}\alpha)=p(\mu^{-1}\sigma^{-1}\alpha)=1 \\ p(\sigma^{-1}\alpha)=-1}} (-1)^{\alpha^\vee} \prod_{\substack{p(\alpha)=p(\sigma^{-1}\alpha)=p(\mu^{-1}\sigma^{-1}\alpha)=1 \\ p(\mu^{-1}\alpha)=-1}} (-1)^{\alpha^\vee}$$

Proof.

$$\begin{aligned}\mu(x(\sigma'))x(\sigma)^{-1} &= \prod_{\substack{p(\alpha)=1 \\ p(\mu^{-1}\sigma^{-1}\mu\alpha)=-1}} (a'_\alpha)^{\mu\alpha^\vee} \prod_{\substack{p(\alpha)=1 \\ p(\sigma^{-1}\alpha)=-1}} a_\alpha^{-\alpha^\vee} \\ &. \\ &= \prod_{\substack{p(\mu^{-1}\alpha)=1 \\ p(\mu^{-1}\sigma^{-1}\alpha)=-1}} a_\alpha^{\alpha^\vee} \prod_{\substack{p(\alpha)=1 \\ p(\sigma^{-1}\alpha)=-1}} a_\alpha^{-\alpha^\vee}.\end{aligned}$$

The contributions to this product are as follows.

(i) $p(\alpha) = 1, p(\mu^{-1}\alpha) = p(\sigma^{-1}\alpha) = -1 : a_\alpha^{-\alpha^\vee}$.

(ii) $p(\alpha) = p(\mu^{-1}\alpha) = p(\mu^{-1}\sigma^{-1}\alpha) = 1, p(\sigma^{-1}\alpha) = -1 : a_\alpha^{-\alpha^\vee}$.

(iii) $p(\mu^{-1}\alpha) = 1, p(\alpha) = p(\sigma^{-1}\alpha) = p(\mu^{-1}\sigma^{-1}\alpha) = -1 : a_\alpha^{\alpha^\vee}$.

(iv) $p(\sigma^{-1}\alpha) = p(\mu^{-1}\alpha) = 1, p(\mu^{-1}\sigma^{-1}\alpha) = -1 : a_\alpha^{\alpha^\vee}$.

On the other hand,

$$\delta\sigma(\delta)^{-1} = \prod_{\substack{p(\alpha)=1 \\ p(\mu^{-1}\alpha)=-1}} a_\alpha^{-\alpha^\vee} \prod_{\substack{p(\sigma^{-1}\alpha)=1 \\ p(\mu^{-1}\sigma^{-1}\alpha)=-1}} a_\alpha^{\alpha^\vee}.$$

The contributions to this product are the following.

(i) $p(\alpha) = 1, p(\mu^{-1}\alpha) = p(\sigma^{-1}\alpha) = -1 : a_\alpha^{-\alpha^\vee}$.

(ii) $p(\alpha) = p(\mu^{-1}\alpha) = p(\mu^{-1}\sigma^{-1}\alpha) = -1, p(\sigma^{-1}\alpha) = 1 : a_\alpha^{\alpha^\vee} = (-1)^{\alpha^\vee} a_{-\alpha}^{\alpha^\vee}$.

(iii) $p(\mu^{-1}\alpha) = -1, p(\alpha) = p(\sigma^{-1}\alpha) = p(\mu^{-1}\sigma^{-1}\alpha) = 1 : a_\alpha^{-\alpha^\vee} = (-1)^{\alpha^\vee} a_{-\alpha}^{-\alpha^\vee}$.

(iv) $p(\sigma^{-1}\alpha) = p(\mu^{-1}\alpha) = 1, p(\mu^{-1}\sigma^{-1}\alpha) = -1 : a_\alpha^{\alpha^\vee}$.

The assertion (a) follows.

For (b) we observe first that $p(\sigma^{-1}\alpha) = p(\alpha)$ so that $p(\omega(\sigma)^{-1}\alpha) = p(\sigma^{-1}\alpha)$ for all roots α. Thus $t_p(\mu, \mu^{-1})\mu(t_p(\mu^{-1}, \omega(\sigma)))$ is equal to

$$\prod_{\substack{p(\alpha)=1\\ p(\mu^{-1}\alpha)=-1}} (-1)^{\alpha^\vee} \prod_{\substack{p(\alpha)=p(\sigma^{-1}\mu\alpha)=1\\ p(\mu\alpha)=-1}} (-1)^{\mu\alpha^\vee} = \prod_{\substack{p(\alpha)=1\\ p(\mu^{-1}\alpha)=-1}} (-1)^{\alpha^\vee} \prod_{\substack{p(\alpha)=1\\ p(\mu^{-1}\alpha)=p(\sigma^{-1}\alpha)=-1}} (-1)^{\alpha^\vee}.$$

Performing the obvious cancellations, we obtain

$$\prod_{\substack{p(\alpha)=p(\sigma^{-1}\alpha)=1\\ p(\mu^{-1}\alpha)=-1}} (-1)^{\alpha^\vee}.$$

Also $\mu(t_p(\mu^{-1}\omega(\sigma), \boldsymbol{\sigma}(\mu))$ is equal to

$$\prod_{\substack{p(\alpha)=p(\mu^{-1}\sigma^{-1}\mu\alpha)=1\\ p(\sigma^{-1}\mu\alpha)=-1}} (-1)^{\mu\alpha^\vee} = \prod_{\substack{p(\mu^{-1}\alpha)=p(\mu^{-1}\sigma^{-1}\alpha)=1\\ p(\sigma^{-1}\alpha)=-1}} (-1)^{\alpha^\vee}.$$

Then $t_p(\mu, \mu^{-1})\mu(t_p(\mu^{-1}\omega(\sigma), \boldsymbol{\sigma}(\mu))t_p(\mu^{-1}\omega(\sigma))$ equals

$$\prod_{\substack{p(\alpha)=p(\sigma^{-1}\alpha)=1\\ p(\mu^{-1}\alpha)=-1}} (-1)^{\alpha^\vee} \prod_{\substack{p(\mu^{-1}\alpha)=p(\mu^{-1}\sigma^{-1}\alpha)=1\\ p(\sigma^{-1}\alpha)=-1}} (-1)^{\alpha^\vee}.$$

The nontrivial contributions to this product occur only for either

(i) $p(\alpha) = p(\mu^{-1}\alpha) = p(\mu^{-1}\sigma^{-1}\alpha) = 1, p(\sigma^{-1}\alpha) = -1$, or (ii) $p(\alpha) = p(\sigma^{-1}\alpha) = p(\mu^{-1}\sigma^{-1}\alpha) = 1, p(\mu^{-1}\alpha) = -1$. Thus (b) follows, and the lemma is proved.

(2.3.4) Suppose now that $\mathrm{Int}\ g^{-1}$ maps T to T' over F and carries a-data $\{a_\alpha\}$ for T to the a-data $\{a'_\alpha\}$ for T'. We construct $\lambda(T) = \lambda_{\{a_\alpha\}}(T)$ and $\lambda(T') = \lambda_{\{a'_\alpha\}}(T')$. Note that if $B \supset T$ is used to define the cocycle $m(\sigma_T)$ then $B' = gBg^{-1}$ yields $m(\sigma_{T'}) = m(\sigma_T)$. From

$$hm(\sigma_T)\sigma(h)^{-1} = g(g^{-1}hm(\sigma_{T'})\sigma(h)^{-1}\sigma(g))g^{-1} \cdot g\sigma(g)^{-1}$$

we conclude that if $\mathbf{g} \in H^1(T)$ is the class of the cocycle $\sigma \to g\sigma(g)^{-1}$ then $\lambda(T)$ is $\mathbf{g}$ times the image of $\lambda(T')$ under the homomorphism $H^1(T') \to H^1(T)$ given by $\mathrm{Int}\ g$.

(2.3.5) Finally, suppose that k is a number field, v is a place of k and k_v is the completion of k at v. Then a-data for T as torus over k, or global a-data serve as well as a-data for T over k_v, or local a-data. We therefore obtain both $\lambda(T) \in H^1(\Gamma, T)$ and $\lambda_v(T) \in H^1(\Gamma_v, T)$ attached to given global a-data and k-splitting of G. For $\sigma \in \Gamma_v \subset \Gamma, m(\sigma_T) = x(\sigma_T)n(\omega_T(\sigma))$ is the same whether given in terms of k or of k_v. Thus $\lambda_v(T)$ is the image of $\lambda(T)$ under $H^1(\Gamma, T) \to H^1(\Gamma_v, T)$.

(2.4) *An Explicit Splitting*

We return to the setting of (2.1). Recall that if p, q are gauges on $\mathcal{R}$ then t_p and t_q are cohomologous 2-cocycles of Σ in $k^\times \otimes X$ (Lemma 2.1.C). Here we shall construct an explicit splitting of t_p/t_q.

Let $s_{p/q}(\sigma) = \prod\limits_{(p)}(-1)^\lambda \prod\limits_{(-q)}(-1)^\lambda, \sigma \in \Gamma$, where $\prod\limits_{(p)}$ is the product over λ such that

$$p(\lambda) = 1, \quad p(\sigma^{-1}\lambda) = -1, \quad q(\lambda) = q(\sigma^{-1}\lambda) = 1$$

and $\prod\limits_{(-q)}$ the product over λ such that

$$p(\lambda) = p(\sigma^{-1}\lambda) = 1, \quad q(\lambda) = -1, \quad q(\sigma^{-1}\lambda) = 1.$$

Observe that $s_{p/p} = s_{-p/p} = s_{p/-p} = 1$.

Lemma 2.4.A. *The coboundary of $s_{p/q}$ is t_p/t_q.*

Proof. For $\sigma, \tau \in \Gamma, \lambda \in \mathcal{R}$, set

$$S(\lambda) = (p(\lambda), p(\sigma^{-1}\lambda), p(\tau^{-1}\sigma^{-1}\lambda), q(\lambda), q(\sigma^{-1}\lambda), q(\tau^{-1}\sigma^{-1}\lambda)).$$

Then

$$s_{p/q}(\sigma)\sigma(s_{p/q}(\tau))s_{p/q}(\sigma\tau)^{-1} = \prod(-1)^\lambda,$$

where the product is taken over the six sets given respectively by $S(\lambda)$ equal to:

$$(1, -1, \pm1, 1, 1, \pm1); (\pm1, 1, -1, \pm1, 1, 1);(1, \pm1, -1, 1, \pm1, 1);$$
$$(1, 1, \pm1, -1, 1, \pm1);(\pm1, 1, 1, \pm1, -1, 1); (1, \pm1, 1, -1, \pm1, 1)$$

. The contribution of $\pm\lambda$ to this product is as follows.

(a) If all entries in $S(\lambda)$ are positive: 1.

(b) If exactly five entries in $S(\lambda)$ are positive: $(-1)^\lambda$ if $S(\lambda) = (1, 1, 1, 1, -1, 1)$ or $(1, -1, 1, 1, 1, 1)$, and 1 otherwise.

(c) If exactly four entries in $S(\lambda)$ are positive: $(-1)^\lambda$ if $S(\lambda) = (1, -1, 1, -1, 1, 1), (1, -1, 1, 1, 1, -1), (1, 1, -1, 1, -1, 1), (-1, 1, 1, 1, -1, 1), (-1, 1, -1, 1, 1, 1)$, or $(1, 1, 1, -1, 1, -1)$, and 1 otherwise.

(d) If exactly three entries in $S(\lambda)$ are positive: $(-1)^\lambda$ if $\pm S(\lambda) = (1,1,-1,-1,1,-1), (1,-1,-1,1,-1,1), (1,-1,1,-1,-1,1)$ or $(-1,1,-1,-1,1,1)$, and 1 otherwise.

On the other hand, $t_p(\sigma,\tau)/t_q(\sigma,\tau) = \prod(-1)^\lambda$ where the product is now over each of

$$\{\lambda : S(\lambda) = (1,-1,1,\pm1,\pm1,\pm1)\}$$

and

$$\{\lambda : S(\lambda) = (\pm1,\pm1,\pm1,1,-1,1)\}.$$

Consider the contribution to this product from λ. In cases (a) and (b) the contribution is the same as for $\partial s_{p/q}$. In cases (c) and (d) if $\pm\lambda$ contribute $(-1)^\lambda$ to $s_{p/q}$ we find that exactly one of λ and $-\lambda$ contributes $(-1)^\lambda$ to t_p/t_q, and conversely. Thus t_p/t_q coincides with $\partial s_{p/q}$, and the lemma is proved.

Corollary 2.4.B.

(i) *$s_{q/p}$ is cohomologous to $s_{p/q}$.*

(ii) *$s_{p/q}s_{q/r}$ is cohomologous to $s_{p/r}$.*

Proof. In view of Lemma 2.4.A both $s_{q/p}s_{p/q}^{-1}$ and $s_{p/q}s_{q/r}s_{p/r}^{-1}$ are cocycles. We can then reduce in the usual way with Shapiro's Lemma to the case $\mathcal{R} = \{\pm\lambda\}$. Then $q, r = \pm p$ and the lemma is clear.

(2.5) χ-*Data*

We consider the case $k = \mathbf{C}$ and $\Gamma = \mathrm{Gal}(L/F)$, where F is a local or a global field and L is a finite Galois extension of F. The group $\Sigma = \langle\Gamma, \epsilon\rangle$ acts on $\mathbf{C}^\times \otimes X$ with trivial action on $\mathbf{C}^\times$. The cocycle t_p is nontrivial in general. Let W be the Weil group of L/F. Then W acts on $\mathbf{C}^\times \otimes X$ through $W \to \Gamma$. The inflation of t_p to W does split [L1]. Here we shall construct a splitting of it using $\chi-data$ which are prescribed as follows.

Set $\Gamma_{+\lambda} = \{\sigma \in \Gamma : \sigma\lambda = \lambda\}$ and $\Gamma_{\pm\lambda} = \{\sigma \in \Gamma : \sigma\lambda = \pm\lambda\}$, $\lambda \in \mathcal{R}$. Once λ has been fixed we delete it in notation. Then $F_+ \subset L$ will be the fixed field of Γ_+, and $F_\pm$ the fixed field of $\Gamma_\pm$. Note that $[F_+ : F_\pm]$ is 2 or 1 according as the Γ-orbit of λ is symmetric or asymmetric. Set $W_\pm = W_{L/F_\pm}$ and $W_+ = W_{L/F_+}$. Then by χ-*data* for the action of Γ on $\mathcal{R}$ we mean a collection $\{\chi_\lambda : \lambda \in \mathcal{R}\}$ such that the following hold.

(i) *χ_λ is a character on $C_{+\lambda}$, where $C_{+\lambda}$ is the multiplicative group of $F_{+\lambda}$ or the idèle-class group of $F_{+\lambda}$, according as F is local or global.*

In either case we may regard χ_λ as a character on $W_+ = W_{+\lambda}$.

(ii) $\chi_{-\lambda} = \chi_\lambda^{-1}$ *and*

$$\chi_{\sigma\lambda} = \chi_\lambda \cdot \sigma^{-1}, \sigma \in \Gamma, \lambda \in \mathcal{R}.$$

(iii) *If* $[F_+ : F_\pm] = 2$ *then* χ_λ, *as character on* C_+, *extends the quadratic character on* $C_\pm$ *attached to the extension* $F_+/F_\pm$. It is readily verified that χ-data exist.

To prescribe the splitting we shall at first assume that Σ acts transitively on $\mathcal{R}$. In general, it is a product of the splittings for the orbits. Fix $\lambda \in \mathcal{R}$ and choose representatives $\sigma_1, \ldots, \sigma_n$ for $\Gamma_\pm \backslash \Gamma$. Then $\pm\sigma_1^{-1}\lambda, \ldots, \pm\sigma_n^{-1}\lambda$ are the elements of $\mathcal{R}$ listed without redundancy. Define a gauge p on $\mathcal{R}$ by

$$p(\lambda') = 1 \text{ if and only if } \lambda' = \sigma_i^{-1}\lambda, \text{ some } 1 \leqq i \leqq n.$$

Choose $w_1, \ldots, w_n \in W$ such that w_i maps to σ_i under $W \to \Gamma$. Then $w_1, \ldots, w_n$ are representatives for $W_\pm \backslash W$. If $w \in W$ then define $u_i(w) \in W_\pm$ by

$$w_i w = u_i(w) w_i, \quad i = 1, \ldots, n.$$

Choose representatives $v_0 \in W_+$ and v_1 for $W_+ \backslash W_\pm$ in case $[F_+ : F_\pm] = 2$ and an element v_0 of W_+ if $F_+ = F_\pm$. Note that $\chi_\lambda(v_1 v v_1^{-1}) = \chi_\lambda(v)^{-1}$. For $u \in W_\pm$ define $v_0(u) \in W_+$ by

$$v_0 \cdot u = v_0(u) \cdot v_{i'},$$

where $i' = 0$ or 1, as appropriate. For $u \in W_\pm$ set

$$s(u) = \chi_\lambda(v_0(u))$$

and for $w \in W$ set

$$r_p(w) = \prod_{i=1}^{n} \chi_\lambda(v_0(u_i(w)))^{\lambda_i} = \prod_{i=1}^{n} s(u_i(w))^{\lambda_i},$$

where $\lambda_i = \sigma_i^{-1}\lambda$. Then r_p is a 1-chain of W with values in $\mathbf{C}^\times \otimes X$.

If q is any gauge on $\mathcal{R}$ set

$$r_q = s_{q/p} r_p.$$

Lemma 2.5.A. *The coboundary of* r_q *is* t_q.

Proof. In view of Lemma 2.4.A we have only to consider the case $q = p$. Suppose that $v, w \in W$. Then

$$w_i v w = u_i(v) w_{i'} w = u_i(v) u_{i'}(w) w_{i''}$$

so that $u_i(vw) = u_i(v) u_{i'}(w)$. If $v \to \sigma$ under $W \to \Gamma$ then

$$\sigma\lambda_{i'} = \sigma\sigma_{i'}^{-1}\lambda = \epsilon_i \sigma_i^{-1}\lambda = \epsilon_i \lambda_i,$$

where $\epsilon_i = +1$ or -1 according as $u_i(v) \in W_+$ or not. Observe that $\epsilon_i = p(\sigma^{-1}\lambda_i)$. Now

$$\begin{aligned} r_p(v)v(r_p(w)) &= \prod_{i=1}^n s(u_i(v))^{\lambda_i} s(u_{i'}(w))^{\sigma\lambda_i} \\ &= \prod_{i=1}^n s(u_i(v))^{\lambda_i} s(u_{i'}(w))^{\epsilon_i\lambda_i} \end{aligned}$$

and

$$r_p(vw) = \prod_{i=1}^n s(u_i(v))^{\lambda_i} s(u_{i'}(w))^{\lambda_i}.$$

We claim that

$$s(u_{i'}(w))^{\epsilon_i} = t \cdot s(u_{i'}(w))$$

where $t = 1$ unless $\epsilon_i = -1$ and $u_{i'}(w) \notin W_+$, in which case it is -1. But $\epsilon_i = p(\sigma^{-1}\lambda_i)$ and if $w \to \tau$ under $W \to \Gamma$ then $u_{i'}(w) \notin W_+$ if and only if $p(\tau^{-1}\lambda_i') = \epsilon_i p(\tau^{-1}\sigma^{-1}\lambda_i) = -1$. We conclude that

$$r_p(v)v(r_p(w))r_p(vw)^{-1} = \prod_{\substack{p(\sigma^{-1}\lambda_i)=-1 \\ p(\tau^{-1}\sigma^{-1}\lambda_i)=1}} (-1)^{\lambda_i} = \prod_{1,\sigma,\tau}^{p} (-1)^{\lambda} = t_p(v,w).$$

To prove the claim there is more convenient notation. Let $u, v \in W_\pm$ and suppose $u \to \sigma, v \to \tau$ under $W_\pm \to \Gamma_\pm$. Set $\epsilon = p(\sigma^{-1}\lambda)$ and $\delta = p(\tau^{-1}\sigma^{-1}\lambda)$. Then the claim asserts that

$$s(u)s(v)^\epsilon s(uv)^{-1} = \begin{cases} -1 & \text{if } \epsilon = -1, \delta = 1 \\ \ \ 1 & \text{otherwise,} \end{cases}$$

where $s(u) = \chi_\lambda(v_0(u))$. If $u \in W_+$, and in particular if $W_+ = W_\pm$, then $\epsilon = 1$ and $v_0(u) = v_0 u v_0^{-1}, v_0(uv) = v_0 u v_0^{-1} v_0(v)$ so that $\chi_\lambda(v_0(u))\chi_\lambda(v_0(v)) = \chi_\lambda(v_0(uv))$. If $u \notin W_+$ then $v_0 uv = v_0(u)v_1 v$. If also $v \in W_+$ then $v_0(u)v_1 v = v_0(u)v_1 v v_1^{-1} v_1$ and $v_0(uv) = v_0(u)v_1 v v_1^{-1}$. Thus

$$\chi_\lambda(v_0(uv)) = \chi_\lambda(v_0(u))\chi_\lambda(v_1 v v_1^{-1}) = \chi_\lambda(v_0(u))\chi_\lambda(v)^{-1}.$$

If both u, v lie outside W_+ then

$$\begin{aligned} v_0 uv = v_0(u)v_1 v &= v_0(u)v_1 v_0^{-1} v_0(v) v_1 \\ &= v_0(u)v_1 v_0^{-1} v_0(v) v_1^{-1} v_1^2 v_0^{-1} v_0. \end{aligned}$$

Thus

$$v_0(uv) = v_0(u)v_1 v_0^{-1} v_0(v) v_1^{-1} v_1^2 v_0^{-1}$$

and

$$\begin{aligned} \chi_\lambda(v_0(uv)) &= \chi_\lambda(v_0(u))\chi_\lambda(v_1 v_0(v) v_1^{-1})\chi_\lambda(v_1 v_0^{-1} v_1^{-1} v_0^{-1})\chi_\lambda(v_1^2) \\ &= -\chi_\lambda(v_0(u))\chi_\lambda(v_0(v))^{-1} \end{aligned}$$

since $\chi_\lambda(v_1^2) = -1$ and $\chi_\lambda(v_1 x v_1^{-1}) = \chi_\lambda(x)^{-1}, x \in W_+$. The claim is thus verified, and Lemma 2.5.A proved.

It remains to check that the various choices made have no effect on r_q, up to 1-coboundaries. First we observe that

$$r_q(x) = r_p(x) = \prod_{i=1}^{n} \chi_\lambda(Nm^L_{F_+}\sigma_i x)^{\sigma_i^{-1}\lambda}, x \in L^\times,$$

is independent of the choices for $v_0, v_1, w_1, \ldots, w_n$ and λ, and that $r_q(xw) = r_q(x)r_q(w), x \in L^\times, w \in W$.

Consider a change in v_0, v_1. Then r_p is replaced by a cochain $r_{p'}$. To show that the cocycle $r_p r_{p'}^{-1}$ is trivial we may assume X is free on $\{\lambda \in \mathcal{R} : p(\lambda) = 1\}$ and reduce to the case $\mathcal{R} = \{\pm\lambda\}$ by Shapiro's Lemma. It is sufficient now to observe that r_p and $r_{p'}$ coincide on W_+, for then $r_p r_{p'}^{-1}$ defines an element of $H^1(\mathrm{Gal}(F_+/F_\pm), \mathbf{C}^\times \otimes X)$. This group is readily seen to be trivial.

Another choice $w_i', \ldots, w_n'$ for $w_1, \ldots, w_n$ leads to a gauge p' and it has to be shown that the cocycle $s_{p'/p} r_p r_{p'}^{-1}$ is trivial. Again we reduce to the case $\mathcal{R} = \{\pm\lambda\}$. Since $p' = \pm p, s_{p'/p}$ is trivial and, as above, it needs only to be shown that r_p and $r_{p'}$ coincide on W_+. We may assume $p(\lambda) = p'(-\lambda) = 1, w_1 = v_0$ and $w_1' = v_1$. Then $\lambda_1 = \lambda, \lambda_1' = -\lambda$ and $u_1(w) = v_0 w v_0^{-1}, u_1'(w) = v_1 w v_1^{-1}, w \in W$. Thus for $w \in W_+$ we have $\chi_{\lambda'}(u_1'(w))^{\lambda_1'} = \chi_\lambda(v_1 w v_1^{-1})^{-\lambda} = \chi_\lambda(w)^\lambda = \chi_\lambda(u_1(w))^\lambda$ and $r_p(w) = r_{p'}(w)$, as desired.

Finally, replacing λ by $-\lambda$ clearly has no effect. If we replace λ by $\lambda' = \varrho\lambda$ and $v \in W$ is a lifting of ϱ then $\Gamma_{+\lambda'} = \varrho\Gamma_{+\lambda}\varrho^{-1}$ and we may take $\sigma_i' = \varrho\sigma_i$. Then $\lambda_i' = \lambda_i$ and we set $w_i' = v w_i$, so that $u_i'(w) = v u_i(w) v^{-1}$. We also take $v_0' = v v_0 v^{-1}$, so that $v_0' u_i'(w) = v v_0 u_i(w) v^{-1}$. Since $\chi_{\lambda'}(v x v^{-1}) = \chi_\lambda(x)$ the independence is clear.

Corollary 2.5.B. *Suppose* $\{\zeta_\lambda : \lambda \in \mathcal{R}\}$ *satisfies:*

(i) ζ_λ *is a character on* $C_{+\lambda}$ *and hence on* $W_{+\lambda}$.

(ii) $\zeta_{\sigma\lambda} = \zeta_\lambda \circ \sigma^{-1}, \sigma \in \Gamma, \lambda \in \mathcal{R}$ *and* $\zeta_{-\lambda} = \zeta_\lambda^{-1}, \lambda \in \mathcal{R}$.

(iii) *If* $[F_{+\lambda} : F_{\pm\lambda}] = 2$ *then* ζ_λ *is trivial on* $C_{\pm\lambda}$.

Then

$$c(w) = \prod_{i=1}^{n} \zeta_\lambda(v_0(u_i(w)))^{\lambda_i} \quad , \quad w \in W \ ,$$

is a 1*-cocycle of* W *with values in* $C^\times \otimes X$*. Its cohomology class is independent of the choices made in its construction.*

If the action of Σ on $\mathcal{R}$ is not transitive we define r_q and c for each Σ-orbit, thus for each pair $\pm\mathcal{O}$ of Γ-orbits, and then take products over all such pairs. The results are denoted again r_q and c.

(2.6) *A Second Application*

Suppose that G is a connected reductive group defined over F. Recall that ${}^L G = \widehat{G} \rtimes W_F$. Suppose that T is a maximal torus over F in G. Then we shall attach to χ-data for T, that is, for the action of $\Gamma = \mathrm{Gal}(\overline{F}/F)$ on $R(G,T)$, a canonical $\widehat{G}$-conjugacy class of admissible embeddings of ${}^L T$ in ${}^L G$.

There will be no harm in replacing $\overline{F}$ throughout by a finite Galois extension $L \subset \overline{F}$ of F over which T splits, and we do so without change in notation. Denote by $\boldsymbol{\sigma}$ the action of $\sigma \in \Gamma$ on $\widehat{G}$ and set $\boldsymbol{\Gamma} = \{\boldsymbol{\sigma} : \sigma \in \Gamma\}$. Fix a Γ-splitting $(\widehat{\mathbf{B}}, \widehat{\mathbf{T}}, \{\mathbf{X}_\alpha^\vee\})$ of $\widehat{G}$.

A homomorphism $\xi : {}^LT \to {}^LG$ is an *admissible embedding* if

(i) ξ maps $\widehat{T}$ to $\widehat{\mathbf{T}}$ by the isomorphism attached to the pair $(\widehat{\mathbf{B}}, \widehat{\mathbf{T}})$ and the choice of a Borel subgroup B in G containing T, and

(ii) $\xi(w) \in \widehat{G} \times w, w \in W$.

The $\widehat{G}$-conjugacy class of ξ is $\{\mathrm{Int}\ g \circ \xi : g \in \widehat{G}\}$. It is independent of the choice of $(\widehat{\mathbf{B}}, \widehat{\mathbf{T}})$ and B.

We fix B. Then to specify ξ we have only to give a homomorphism $w \to \xi(w) = \xi_0(w) \times w$ where $\xi_0(w) \in \mathrm{Norm}(\widehat{\mathbf{T}}, \widehat{G})$ and where, in addition, if $w \to \sigma$ under $W \to \Gamma$ then Int $\xi(w)$ acts on $\widehat{\mathbf{T}}$ as the transport by ξ of the action of $\sigma \in \Gamma$ on $\widehat{T}$. We write this transport σ_T as $\omega_T(\sigma) \rtimes \boldsymbol{\sigma} \in \boldsymbol{\Omega} \times \boldsymbol{\Gamma}$, and set $\Gamma_T = \{\sigma_T : \sigma \in \Gamma\}$. The χ-data $\{\chi_\alpha\}$ for T provide χ-data for the action of Γ_T on $\mathcal{R} = R^\vee(\widehat{G}, \widehat{\mathbf{T}})$; $\{\chi_\alpha^{-1}\}$ is also a set of χ-data.

By Lemma 2.1.A

$$n(w) = n(\omega_T(\sigma)) \times w \in \ \mathrm{Norm}(\widehat{\mathbf{T}}, \widehat{G}) \times w, \quad w \in W,$$

satisfies

$$n(w_1)n(w_2)n(w_1w_2)^{-1} = t_p(\sigma_1, \sigma_2),$$

if $w_i \to \sigma_i$ under $W \to \Gamma$, where $p(\alpha) = 1$ if and only if $\alpha^\vee$ is a root of $\widehat{\mathbf{T}}$ in $\widehat{\mathbf{B}}$. By Lemma 2.5.A the inflation of t_p to W is split by r_p^{-1}, where r_p is the 1-cochain of W in $\widehat{\mathbf{T}}$ attached to $\{\chi_\alpha\}$ as in (2.5). We now use p_0 for the gauge fixed there so that $r_p = s_{p/p_0} r_{p_0}$, and note that r_p^{-1} is the cochain attached to $\{\chi_\alpha^{-1}\}$. Thus with

$$\xi(w) = r_p(w) n(\omega_T(\sigma)) \times w, \quad w \in W$$

we obtain an admissible homomorphism $\xi : {}^LT \to {}^LG$. It is determined uniquely up to $\widehat{\mathbf{T}}$-conjugacy by the Γ-splitting, Borel subgroup B and χ-data for T.

(2.6.1) Suppose that the Γ-splitting is replaced by another, $(\mathbf{B}^g, \mathbf{T}^g, \{\mathbf{X}_{\alpha^\vee}^g\})$. We may suppose that $g \in \widehat{G}$ is $\boldsymbol{\Gamma}$-invariant [K2]. Then $n(w)$ is replaced by $g^{-1}n(w)g, w \in W$, and $\xi : {}^LT \to {}^LG$ by Int $g^{-1} \cdot \xi$, so that the $\widehat{G}$-conjugacy class of ξ is not affected.

(2.6.2) We show now that the $\widehat{G}$-conjugacy class of ξ is independent of the choice of B. Suppose that B is replaced by $B' = vBv^{-1}$, where $v \in \mathrm{Norm}(T, G)$, and ξ' is obtained in place of ξ. Let μ in $\boldsymbol{\Omega} = \Omega(\widehat{G}, \widehat{\mathbf{T}})$ be the element defined by the transport of Int $v|_T$ to $\widehat{\mathbf{T}}$ by ξ. Then the transport of σ from T to $\widehat{\mathbf{T}}$ via ξ' is $\sigma_T' = \omega_T'(\sigma) \rtimes \boldsymbol{\sigma}$, where $\omega_T'(\sigma) = \mu^{-1}\omega_T(\sigma)\boldsymbol{\sigma}(\mu)$.

Lemma 2.6.A. *We have*

$$\xi' = \text{Int } g^{-1} \circ \xi,$$

where $g \in$ Norm$(\widehat{\mathbf{T}}, \widehat{G})$ *acts on* $\widehat{\mathbf{T}}$ *as* μ.

Proof. The two sets of data $\{\chi_\alpha\}$ and $\{\chi'_\alpha = \chi_{\mu\alpha}\}$ yield

$$\xi(w) = r_p(w) n(\omega_T(\sigma)) \times w, \quad \xi'(w) = r'_p(w) n(\omega'_T(\sigma)) \times w$$

if $w \to \sigma$ under $W \to \Gamma$. We delete the subscript T in our notation. Lemma 2.1.A shows that

$$\begin{aligned}
\xi'(w) &= r'_p(w) n(\mu^{-1}\omega(\sigma)\boldsymbol{\sigma}(\mu)) \times w \\
&= r'_p(w) t_p(\mu^{-1}\omega(\sigma), \boldsymbol{\sigma}(\mu)) t_0(\mu^{-1}, \omega(\sigma)) n(\mu^{-1}) n(\omega(\sigma)) n(\boldsymbol{\sigma}(\mu)) \times w \\
&= n(\mu)^{-1} t_p(\mu, \mu^{-1}) \mu[t_p(\mu^{-1}\omega(\sigma), \boldsymbol{\sigma}(\mu)) t_p(\mu^{-1}, \omega(\sigma)) r'_p(w)] r_p(w)^{-1} \xi(w) n(\mu).
\end{aligned}$$

By Lemma 2.3.B,

$$t_p(\mu, \mu^{-1}) \mu[t_p(\mu^{-1}\omega(\sigma), \boldsymbol{\sigma}(\mu)) t_p(\mu^{-1}, \omega(\sigma))]$$

is equal to

$$\prod_{\substack{p(\alpha)=p(\mu^{-1}\alpha)=p(\mu^{-1}\sigma^{-1}\alpha)=1 \\ p(\sigma^{-1}\alpha)=-1}} (-1)^\alpha \prod_{\substack{p(\alpha)=p(\sigma^{-1}\alpha)=p(\mu^{-1}\sigma^{-1}\alpha)=1 \\ p(\mu^{-1}\alpha)=-1}} (-1)^\alpha = s_{q/p}(\sigma),$$

where q is the gauge $p \circ \mu^{-1}$.

To complete the proof we observe that $\mu(r'_p(w)) = s_{q/p_0}(\sigma) r_{p_0}(w)$, so that

$$\begin{aligned}
\mu(r'_p(w)) r_p(w)^{-1} &= s_{q/p_0}(\sigma) r_{p_0}(w) s_{p/p_0}(\sigma)^{-1} r_{p_0}(w)^{-1} \\
&= s_{q/p}(\sigma).
\end{aligned}$$

(2.6.3) Suppose that the χ-data $\{\chi_\alpha\}$ are replaced by $\{\chi'_\alpha\}$. Then $\chi'_\alpha = \zeta_\alpha \chi_\alpha$, where $\{\zeta_\alpha : \alpha \in \mathcal{R}\}$ satisfies the conditions of Corollary 2.5.B. Let c be the cocycle defined there. Then the embedding ξ is replaced by $c \otimes \xi$ where

$$c \otimes \xi(t \times w) = c(w) \xi(t \times w), \quad t \times w \in {}^L T.$$

(2.6.4) Suppose that Int g^{-1} maps T to T' over F and carries the χ-data $\{\chi_\alpha\}$ for T to data $\{\chi'_\alpha\}$ for T'. Now g defines a canonical isomorphism $\lambda_g : {}^L T' \to {}^L T$. Let ξ be the embedding of ${}^L T$ in ${}^L G$ defined (up to $\xi(\widehat{T})$-conjugacy) by the choice of Borel subgroup $B \supset T$. Then $\xi' = \xi \circ \lambda_g$ is the embedding of ${}^L T'$ in ${}^L G$ defined by the Borel subgroup $B' = g^{-1} B g$ of G. Thus the class of embeddings attached to $(T', \{\chi'_\alpha\})$ is obtained from that attached to $(T, \{\chi_\alpha\})$ by composition with the canonical ${}^L T' \to {}^L T$.

(2.6.5) There is a simple local-global relationship when F is a number field. Suppose that v is a place of F and F_v is the completion of F at v. Fix a place w of L dividing v, and set $\Gamma_v = \mathrm{Gal}(L_w/F_v), W_v = W_{L_w/F_v}$. Then a homomorphism $W_v \to W$ for which

$$\begin{array}{ccccccccc} 1 \to & L_w^\times \to & W_v \to & \Gamma_v \to & 1 \\ & \downarrow & \downarrow & \downarrow & \\ 1 \to & C_{L_w} \to & W \to & \Gamma \to & 1 \end{array}$$

is commutative attaches to $\xi : \widehat{T} \rtimes W \to \widehat{G} \rtimes W$ a local embedding $\xi_v : \widehat{T} \rtimes W_v \to \widehat{G} \rtimes W_v$. Since $\xi(C_{L_w}) \subset \widehat{\mathbf{T}} \times C_{L_w}$, where $\widehat{\mathbf{T}} = \xi(\widehat{T}), \xi_v$ is determined uniquely up to $\widehat{\mathbf{T}}$-conjugacy by ξ.

On the other hand, the place w of L determines completions of the subfields, F_+ and $F_\pm$. These completions coincide with $F_{v,+}$ and $F_{v,\pm}$, the subfields of L_w defined by $\Gamma_{v,+} = \{\sigma \in \Gamma_v : \sigma\lambda = \lambda\}$ and $\Gamma_{v,\pm} = \{\sigma \in \Gamma_v : \sigma\lambda = \pm\lambda\}$. Then the natural embeddings $F_{v,+} \hookrightarrow C_+, F_{v,\pm} \hookrightarrow C_\pm$ allow us to construct χ-data $\{\chi_\alpha^{(v)}\}$ for the action of Γ_v, local data, from χ-data $\{\chi_\alpha\}$ for the action of Γ, global data.

To prescribe an admissible $\xi : \widehat{T} \rtimes W \to \widehat{G} \rtimes W$ we have to give a Γ-splitting of $\widehat{G}$ and a Borel subgroup B of G containing T as well as $\{\chi_\alpha\}$. The same splitting and Borel subgroup together with $\{\chi_\alpha^{(v)}\}$ yield a local embedding $\xi'_v : \widehat{T} \rtimes W_v \to \widehat{G} \rtimes W_v$. Because the choices made in our constructions do not matter we can arrange that $\xi'_v(w_v) = \xi(w)$ if $w_v \to w$ under $W_v \to W$. Thus, up to $\widehat{\mathbf{T}}$-conjugacy, ξ'_v is the local embedding attached to ξ, and passage from global to local embeddings is consistent with passage from global to local χ-data.

3. Definitions

(3.1) *Notation*

Throughout Sect. 3 F will be local and $\gamma_H, \overline{\gamma}_H$ will be strongly G-regular elements in $H(F)$ which are images of the elements $\gamma_G, \overline{\gamma}_G$ in $G(F)$.

Let $T_H, \overline{T}_H$, be the centralizers of $\gamma_H, \overline{\gamma}_H$ in H. We fix admissible embeddings $T_H \to T$ and $\overline{T}_H \to \overline{T}$ of T_H and $\overline{T}_H$ *in* G^*, *the quasi-split form of* G, and denote by γ and $\overline{\gamma}$ the images of γ_H and $\overline{\gamma}_H$ under these embeddings. Notation for tori and elements in G itself will always include the subscript G.

We denote by R the root system of T in G^*, by $R^\vee$ the coroots, by Ω the Weyl group, by $R_H^\vee$ the subsystem of coroots from H, by R_H the subset of roots from H and by Ω_H the Weyl group generated by R_H or $R_H^\vee$. The analogous objects for $\overline{T}$ will be denoted $\overline{R}, \overline{R}^\vee$ and so on.

We further fix a-data and χ-data for the action of $\Gamma = \mathrm{Gal}(\overline{F}/F)$ on the roots of T and of $\overline{T}$. These may also be regarded as a-data and χ-data for the action on the coroots, and are unaffected by passage to the simply-connected covering G^*_{sc} of the derived group G^*. If $T_H \to T$ is replaced by an $\mathfrak{A}(T)$-conjugate [recall (1.3)] then we may use this conjugation to transport given a-data and χ-data to data for the new image of T_H.

To check the effect of our choice of embedding $T_H \to T$, a-data $\{a_\alpha\}$ and χ-data $\{\chi_\alpha\}$ we will have only to determine the effect of:

(A) *replacing* $(T_H \to T, \{a_\alpha\}, \{\chi_\alpha\})$ *by an* $\mathfrak{A}(T)$*-conjugate triple, and*

(B) *changing* $\{a_\alpha\}, \{\chi_\alpha\}$ *with* $T_H \to T$ *fixed.*

Recall that $T_H \to T$ is given by the choice of Borel subgroups B_H in H and B in G^*, of pairs $(\mathcal{B}_H, \mathcal{T}_H)$ in $\widehat{H}$ and $(\mathcal{B}, \mathcal{T})$ in $\widehat{G}$, and of $x \in \widehat{G}$ such that $\operatorname{Int} x \circ \xi$ maps $\mathcal{T}_H$ to $\mathcal{T}$ and $\mathcal{B}_H$ into $\mathcal{B}$. Several choices yield the same embedding. We may, without loss of generality, fix Γ-splittings $(\mathcal{B}_H, \mathcal{T}_H, \{X^H\})$ of $\widehat{H}$ and $(\mathcal{B}, \mathcal{T}, \{X\})$ of $\widehat{G}$ and require that $(\mathcal{B}_H, \mathcal{T}_H)$ and $(\mathcal{B}, \mathcal{T})$ be the chosen pairs in $\widehat{H}$ and $\widehat{G}$ for both $T_H \to T$ and $\overline{T}_H \to \overline{T}$. Up to equivalence of endoscopic data we may assume that ξ maps $\mathcal{T}_H$ to $\mathcal{T}$ and $\mathcal{B}_H$ into $\mathcal{B}$. Since the endoscopic datum s is central in $\xi(\widehat{H})$, it lies in $\mathcal{T}$ and its preimage in $\widehat{T}_H$ is independent of the choice of B_H; so its image s_T in $\widehat{T}$ depends only on the embedding $T_H \to T$.

There is a canonical embedding of the center $Z(\widehat{G})$ of $\widehat{G}$ in $\widehat{T}$. Let $\widehat{T}_{\mathrm{ad}} = \widehat{T}/Z(\widehat{G})$. Then $\pi_0 = \pi_0(\widehat{T}_{\mathrm{ad}}^\Gamma)$ will denote the component group of the Γ-invariants in $\widehat{T}_{\mathrm{ad}}$. The image of s_T in $\widehat{T}_{\mathrm{ad}}$ is Γ-invariant and so defines an element s_T of π_0. Finally we recall from [K2] that Tate-Nakayama duality provides a pairing

$$\langle , \rangle : H^1(\Gamma, T_{\mathrm{sc}}) \times \pi_0 \to \mathbf{C}^\times.$$

(3.2) *Term* Δ_I

Definition.

$$\Delta_I(\gamma_H, \gamma_G) = \langle \lambda(T_{\mathrm{sc}}), \mathrm{s}_T \rangle,$$

where $\lambda(T_{\mathrm{sc}})$ is computed relative to an F-splitting **spl** of G_{sc} [see (2.3)].

Lemma 3.2.A.

$$\Delta_I(\gamma_H, \gamma_G)/\Delta_I(\overline{\gamma}_H, \overline{\gamma}_G)$$

is independent of the choice of **spl**.

Proof. Suppose **spl** is replaced by $\mathbf{spl}^g$, where $g \in G^*_{\mathrm{sc}}(\overline{F})$ is such that $g\sigma(g^{-1})$ lies in the center Z_{sc} of G^*_{sc}, $\sigma \in \Gamma$. Then $\lambda(T_{\mathrm{sc}})$ is multiplied by the class $\mathbf{g}_T$ of $\sigma \to g\sigma(g^{-1})$ in $H^1(T^*_{\mathrm{sc}})$, by (2.3.1). Thus we have to show that

$$\langle \mathbf{g}_T, \mathrm{s}_T \rangle = \langle \mathbf{g}_{\overline{T}}, \mathrm{s}_{\overline{T}} \rangle.$$

There will be no harm in replacing $\overline{F}$ by a finite Galois extension over which T is split. We do so without change in notation. Then following [L2, Lemma 6.2] we identify $H^1(T_{\mathrm{sc}})$ with $H^1(X_*(T_{\mathrm{sc}})), H^1(Z_{\mathrm{sc}})$ with $H^{-2}(X_*(T_{\mathrm{ad}})/X_*(T_{\mathrm{sc}}))$ and $H^1(Z_{\mathrm{sc}}) \to H^1(T_{\mathrm{sc}})$ with

$$H^{-2}(X_*(T_{\mathrm{ad}})/X_*(T_{\mathrm{sc}})) \to H^{-1}(X_*(T_{\mathrm{sc}})).$$

Suppose the class of $\sigma \to g\sigma(g^{-1})$ in $H^1(Z_{\mathrm{sc}})$ corresponds to that of $\sigma \to \boldsymbol{\lambda}_\sigma$, where $\boldsymbol{\lambda}_\sigma \in X_*(T_{\mathrm{ad}})/X_*(T_{\mathrm{sc}})$. Let λ_σ be a coset representative for $\boldsymbol{\lambda}_\sigma$. Then $\mathbf{g}_T$ corresponds to the class of $\Sigma_\sigma(\sigma_T^{-1} - 1)\lambda_\sigma$ in $H^{-1}(X_*(T_{\mathrm{sc}}))$ and $\langle \mathbf{g}_T, \mathbf{s}_T \rangle = (\Sigma_\sigma(\sigma_T^{-1} - 1)\lambda_\sigma)(s_T)$ if we regard the sum as a character on $\widehat{T}$. Similarly, there is a formula

$$\langle \mathbf{g}_{\bar{T}}, \mathbf{s}_{\bar{T}} \rangle = (\Sigma_\sigma(\sigma_{\overline{T}}^{-1} - 1)\overline{\lambda}_\sigma)(s\overline{T}).$$

To compare the two, we may choose an element x of G^* such that $x^{-1}Tx = \overline{T}$ and such that $t \to x^{-1}tx$ is the transfer to $T, \overline{T}$ of an isomorphism $T_H \to \overline{T}_H$ inner in H. Then $x\sigma(x^{-1})$ represents an element of Ω_H. Under $\mathrm{Int}\ x^{-1}$, $\overline{T}$ is identified as T with twisted Galois action $\sigma_{\overline{T}} = \varrho(\sigma) \rtimes \sigma_T$, where $\varrho(\sigma) \in \Omega_H$, and $s_{\overline{T}}$ is identified with s_T. We may assume $\overline{\lambda}_\sigma = \lambda_\sigma$, so that

$$\begin{aligned}
\langle \mathbf{g}_{\overline{T}}, s_{\overline{T}} \rangle &= \left(\sum_\sigma (\sigma_T^{-1}\varrho(\sigma)^{-1} - 1)\lambda_\sigma \right)(s_T) \\
&= \left(\sum_\sigma (\sigma_T^{-1} - 1)\lambda_\sigma \right)(s_T) \\
&= \langle \mathbf{g}_T, s_T \rangle,
\end{aligned}$$

since $\varrho(\sigma) \in \Omega_H$ fixes $\sigma_T(s_T)$.

Lemma 3.2.B. *If $(T_H \to T, \{a_\alpha\})$ is replaced by its g-conjugate, $g \in \mathfrak{A}(T_{\mathrm{sc}})$, then $\Delta_I(\gamma_H, \gamma_G)$ is multiplied by*

$$\langle \mathbf{g}_T, \mathbf{s}_T \rangle^{-1},$$

where $\mathbf{g}_T$ is the class of $\sigma \to g\sigma(g^{-1})$ in $H^1(T_{\mathrm{sc}})$.

Proof. This follows immediately from (2.3.4).

Lemma 3.2.C. *Suppose that the a-data $\{a_\alpha\}$ are replaced by $\{a'_\alpha\}$. Set $b_\alpha = a'_\alpha/a_\alpha$. Then $\Delta_I(\gamma_H, \gamma_G)$ is multiplied by:*

$$\prod_\alpha \chi_\alpha(b_\alpha),$$

where $\{\chi_\alpha\}$ are χ-data and the product is over representatives α for the symmetric orbits of Γ in the roots of T that are outside H.

Recall that $b_\alpha \in F_{\pm\alpha}^\times$ and that if α belongs to a symmetric orbit then χ_α restricts to the quadratic character on $F_{\pm\alpha}^\times$ attached to $F_{+\alpha}/F_{\pm\alpha}$ (2.5). Also $\chi_\alpha(b_\alpha) = \chi_{\sigma\alpha}(b_{\sigma\alpha}), \sigma \in \Gamma$, so that the choice of orbit representative does not matter.

Proof of the lemma. By (2.3.2) $\Delta_I(\gamma_H, \gamma)$ is multiplied by $\langle \mathbf{b}, \mathbf{s}_T \rangle$ where $\mathbf{b}$ is represented by the cocycle $\sigma \to \prod_{1,\sigma}^{q} b_\alpha^{\alpha^\vee}$, q being some gauge on R. The choice of q does not matter (Lemma 2.2.C). If $\mathcal{O}$ is a Γ-orbit then the contribution from $\pm\mathcal{O}$ to this product is also a cocycle. Suppose it represents the class $\mathbf{b}_{\pm\mathcal{O}}$. Then $\mathbf{b} = \prod_{\pm\mathcal{O}} \mathbf{b}_{\pm\mathcal{O}}$. If $\mathcal{O}$ is asymmetric then $\mathbf{b}_{\pm\mathcal{O}}$ is trivial (Lemma 2.2.C). Thus it remains to show the following:

Lemma 3.2.D. *If $\mathcal{O}$ is symmetric then*

(i) $\langle \mathbf{b}_{\mathcal{O}}, \mathbf{s}_T \rangle = 1$ *if $\mathcal{O}$ is contained in R_H and*

(ii) $\langle \mathbf{b}_{\mathcal{O}}, \mathbf{s}_T \rangle = \chi_\alpha(b_\alpha)$ *if $\mathcal{O}$ is outside R_H, where α represents $\mathcal{O}$.*

Proof. We extend the Shapiro lemma arguments of Sect. 2. Let $X_{\mathcal{O}}$ be the free abelian group on $\mathcal{O}_+^\vee = \{\alpha^\vee : \alpha \in \mathcal{O}, q(\alpha) = 1\}$ with the inherited action of Γ. Fix some $\alpha^\vee \in \mathcal{O}_+^\vee$ and let X_α be the subgroup generated by $\alpha^\vee$. Then the stabilizer $\Gamma_{\pm\alpha}$ of $\{\pm\alpha\}$ in Γ acts on X_α and $X_{\mathcal{O}} = \mathrm{Ind}_{\Gamma_{\pm\alpha}}^{\Gamma} X_\alpha$. Let $T_{\mathcal{O}}$ be the torus over F with $X_*(T_{\mathcal{O}}) = X_{\mathcal{O}}$ and T_α be the torus over $F_{\pm\alpha}$ with $X_*(T_\alpha) = X_\alpha$. Then T_α is one-dimensional, anisotropic over $F_{\pm\alpha}$ and split over F_α, and $T_{\mathcal{O}} = \mathrm{Res}_F^{F_{\pm\alpha}} T_\alpha$. From the natural homomorphism $X_{\mathcal{O}} \to X_*(T_{\mathrm{sc}})$ we obtain $T_{\mathcal{O}} \to T_{\mathrm{sc}}$ over F and an L-homomorphism $\widehat{T}_{\mathrm{ad}} \to \widehat{T}_{\mathcal{O}}$. Let $s_{\mathcal{O}}$ be the image of s_T, or more precisely of the image of s_T in $\widehat{G}_{\mathrm{ad}}$. Then

$$\alpha^\vee(s_{\mathcal{O}}) = \alpha^\vee(s_T), \quad \alpha \in \mathcal{O}.$$

We pull $\mathbf{b}_{\mathcal{O}}$ back to $H^1(T_{\mathcal{O}})$, as we may, without change in notation. If $\mathbf{s}_{\mathcal{O}}$ is the image of $\mathbf{s}_T$ under $\pi_0(\widehat{T}_{\mathrm{ad}}^\Gamma) \to \pi_0(\widehat{T}_{\mathcal{O}}^\Gamma)$ then the functoriality of Tate-Nakayama duality yields

$$\langle \mathbf{b}_T, \mathbf{s}_T \rangle = \langle \mathbf{b}_{\mathcal{O}}, \mathbf{s}_{\mathcal{O}} \rangle.$$

If $\mathcal{O} \subset R_H$ then $\alpha^\vee(s_T) = 1, \alpha \in \mathcal{O}$, and so $\mathbf{s}_{\mathcal{O}} = 1$. This proves the first assertion of the lemma.

If $\mathcal{O}$ is outside H we compute $\langle \mathbf{b}_{\mathcal{O}}, \mathbf{s}_{\mathcal{O}} \rangle$ by reduction to T_α. The image $\mathbf{b}_\alpha$ of $\mathbf{b}_{\mathcal{O}}$ under $H^1(\Gamma, T_{\mathcal{O}}) \cong H^1(\Gamma_{\pm\alpha}, T_\alpha)$ is represented by

$$\sigma \to \begin{cases} b^{\alpha^\vee}, & \sigma\alpha = -\alpha \\ 1, & \sigma\alpha = \alpha \end{cases}.$$

Let $\mathbf{s}_\alpha$ be the image of $\mathbf{s}_{\mathcal{O}}$ under $\pi_0(\widehat{T}_{\mathcal{O}}^\Gamma) \cong \pi_0(\widehat{T}_\alpha^{\Gamma_{\pm\alpha}})$. Then by restriction of scalars for Tate-Nakayama duality we have:

$$\langle \mathbf{b}_{\mathcal{O}}, \mathbf{s}_{\mathcal{O}} \rangle = \langle \mathbf{b}_\alpha, \mathbf{s}_\alpha \rangle.$$

Since $\pi_0(\widehat{T}_\alpha^{\Gamma_{\pm\alpha}})$ consists of two elements and $\mathbf{s}_{\mathcal{O}}$ is clearly nontrivial $\mathbf{s}_\alpha$ must be nontrivial. Then, as observed in (1.1),

$$\langle \mathbf{b}_\alpha, \mathbf{s}_\alpha \rangle = \chi_\alpha(b_\alpha),$$

and the lemma is proved.

(3.3) *Term* Δ_{II}

Definition. $\Delta_{II}(\gamma_H, \gamma_G) = \prod \chi_\alpha\left(\frac{\alpha(\gamma)-1}{a_\alpha}\right)$, where the product is over representatives α for the orbits of Γ in the roots of T that are outside H.

Since

$$\chi_{\sigma\alpha}\left(\frac{\sigma\alpha(\gamma)-1}{a_{\sigma\alpha}}\right) = \chi_\alpha \circ \sigma^{-1}\left(\frac{\sigma(\alpha(\gamma))-1}{\sigma(a_\alpha)}\right) = \chi_\alpha\left(\frac{\alpha(\gamma)-1}{a_\alpha}\right)$$

the choice of representative α does not matter.

Lemma 3.3.A. *If $\mathcal{O}$ is asymmetric then the contribution from $\pm\mathcal{O}$ is $\chi_\alpha(\alpha(\gamma))$, where α lies in either $\mathcal{O}$ or $-\mathcal{O}$.*

Proof.

$$\chi_\alpha\left(\frac{\alpha(\gamma)-1}{a_\alpha}\right)\chi_{-\alpha}\left(\frac{\alpha(\gamma)^{-1}-1}{a_{-\alpha}}\right) = \chi_\alpha\left(\frac{\alpha(\gamma)-1}{a_\alpha}\right)\chi_\alpha\left(\frac{-a_\alpha}{\alpha(\gamma)^{-1}-1}\right) = \chi_\alpha(\alpha(\gamma)).$$

Lemma 3.3.B. *If $(T_H \to T, \{a_\alpha\}, \{\chi_\alpha\})$ is replaced by an $\mathfrak{A}(T)$-conjugate then $\Delta_{II}(\gamma_H, \gamma_G)$ is unchanged.*

Proof. This is immediate.

Lemma 3.3.C. *If the a-data $\{a_\alpha\}$ are replaced by $\{a'_\alpha\}$, where $a'_\alpha = a_\alpha b_\alpha$, then $\Delta_{II}(\gamma_H, \gamma_G)$ is multiplied by*

$$\prod_\alpha \chi_\alpha(b_\alpha)^{-1},$$

where the product is over representatives for the symmetric orbits outside H.

Proof. By definition $\Delta_{II}(\gamma_H, \gamma_G)$ is multiplied by $\prod \chi_\alpha(b_\alpha)^{-1}$, where the product is over representatives for all orbits outside H. Since

$$\chi_\alpha(b_\alpha)\chi_{-\alpha}(b_{-\alpha}) = \chi_\alpha(b_\alpha)\chi_\alpha^{-1}(b_\alpha) = 1$$

we may ignore the asymmetric orbits.

It remains to consider the effect on $\Delta_{II}(\gamma_H, \gamma_G)$ of replacing the χ-data $\{\chi_\alpha\}$ by $\{\chi'_\alpha\}$. Suppose $\zeta_\alpha = \chi'_\alpha\chi_\alpha^{-1}$. Then if α lies in a symmetric orbit ζ_α must be an extension to $F_\alpha^\times$ of the trivial character on $F_{\pm\alpha}^\times$.

Suppose that $\mathcal{O}$ is symmetric and q is some gauge on $\mathcal{O}$. We denote by $X^{\mathcal{O}}$ the free abelian group on $\mathcal{O}_+ = \{\alpha \in \mathcal{O} : q(\alpha) = 1\}$ with the inherited action of Γ, and let X^α be the submodule generated by some $\alpha \in \mathcal{O}_+$, so that $X^{\mathcal{O}} = \mathrm{Ind}_{\Gamma_{\pm\alpha}}^{\Gamma} X^\alpha$. Define the torus $T^{\mathcal{O}}$ over F by $X^*(T^{\mathcal{O}}) = X^{\mathcal{O}}$ and T^α over $F_{\pm\alpha}$ by $X^*(T^\alpha) = X^\alpha$. Then T^α is one-dimensional, anisotropic over $F_{\pm\alpha}$ and split over F_α, and $T^{\mathcal{O}} = \mathrm{Res}_F^{F_{\pm\alpha}} T^\alpha$. From the natural homomorphism $X^{\mathcal{O}} \to X^*(T)$ we obtain a homomorphism $T \to T^{\mathcal{O}}$ over F and then

$$T(F) \to T^{\mathcal{O}}(F) \xrightarrow{\sim} T^\alpha(F_{\pm\alpha}).$$

Let γ^α be the image of γ in $T^\alpha(F_{\pm\alpha})$. Then

$$\alpha(\gamma) = \alpha(\gamma^\alpha).$$

Since the norm map $T^\alpha(F_\alpha) \to T^\alpha(F_{\pm\alpha})$ is surjective we may write

$$\gamma^\alpha = \delta^\alpha \overline{\delta^\alpha},$$

where $\delta^\alpha \in T^\alpha(F_\alpha)$ and the bar denotes conjugation in $T^\alpha(F_\alpha)$ with respect to $T^\alpha(F_{\pm\alpha})$.

Although we could do without it here we describe the analogous construction for an asymmetric orbit $\mathcal{O}$. Thus $X^{\pm\mathcal{O}}$ is the free abelian group on $\mathcal{O}$ and X^α is the subgroup generated by some α in $\mathcal{O}$. Then Γ acts and X^α has stabilizer $\Gamma_{+\alpha} = \Gamma_{\pm\alpha}$. We define $T^{\pm\mathcal{O}}$ over F by $X^*(T^{\pm\mathcal{O}}) = X^{\pm\mathcal{O}}$; T^α is the one-dimensional F_α-split torus with $X^*(T^\alpha) = X^\alpha$, and $T^{\pm\mathcal{O}} = \mathrm{Res}^F_{F_\alpha} T^\alpha$. From $X^{\pm\mathcal{O}} \to X^*(T)$ we obtain

$$T(F) \to T^{\pm\mathcal{O}}(F) \xrightarrow{\sim} T^\alpha(F_\alpha).$$

If γ^α is the image of γ then $\alpha(\gamma^\alpha) = \alpha(\gamma)$.

Lemma 3.3.D. *If the χ-data $\{\chi_\alpha\}$ are replaced by $\{\chi'_\alpha\}$, where $\chi'_\alpha = \chi_\alpha\zeta_\alpha$, then $\Delta_{II}(\gamma_H, \gamma_G)$ is multiplied by*

$$\prod^{\text{asymm}} \zeta_\alpha(\gamma^\alpha) \prod^{\text{symm}} \zeta_\alpha(\delta^\alpha),$$

where the product $\prod^{\text{asymm}}$ is over representatives α for pairs $\pm\mathcal{O}$ of asymmetric orbits outside H, and $\prod^{\text{symm}}$ is over representatives for the symmetric orbits outside H.

Proof. The contribution $\prod^{\text{asymm}}$ is clear from Lemma 3.3.A. If α lies in a symmetric orbit we have

$$\alpha(\gamma) = \alpha(\gamma^\alpha) = \alpha(\delta^\alpha\overline{\delta^\alpha}) = \alpha(\delta^\alpha)\Big/\overline{\alpha(\delta^\alpha)}.$$

Hence

$$\begin{aligned} \zeta_\alpha\left(\frac{\alpha(\gamma)-1}{a_\alpha}\right) &= \zeta_\alpha\left(\frac{\alpha(\delta^\alpha) - \overline{\alpha(\delta^\alpha)}}{a_\alpha\overline{\alpha(\delta^\alpha)}}\right) \\ &= \zeta_\alpha\left(\frac{\alpha(\delta^\alpha) - \alpha(\delta^\alpha)}{a_\alpha}\right) \cdot \zeta_\alpha\overline{(\alpha(\delta^\alpha))}^{-1} \\ &= \zeta_\alpha(\alpha(\delta^\alpha)) \end{aligned}$$

since $(\alpha(\delta^\alpha) - \overline{\alpha(\delta^\alpha))}/a_\alpha$ lies in $F^\times_{\pm\alpha}$ and $\zeta_\alpha\overline{(\alpha(\delta^\alpha))}^{-1} = \zeta_\alpha(\alpha(\delta^\alpha))$.

(3.4) *Term Δ_{III_1} or Δ_1*

The next two terms will be denoted Δ_{III_1} and Δ_{III_2}, or more briefly Δ_1 and Δ_2, since they are combined in a single term in the twisted case [K-S].

We begin with the case that G is quasi-split over F, taking G as G^* and the twist ψ to be the identity. Since γ_H is an image of γ_G there exists $h \in G_{\text{sc}}$ such that $g\gamma_G h^{-1} = \gamma$. Set $v(\sigma) = h\sigma(h)^{-1}$. Because γ is strongly regular the class $\mathrm{inv}(\gamma_H, \gamma_G)$ of $v(\sigma)$ in $H^1(T_{\text{sc}})$ is well-defined.

Definition (G quasi-split). $\Delta_1(\gamma_H, \gamma_G) = \langle \mathrm{inv}(\gamma_H, \gamma_G)\mathbf{s}_T\rangle^{-1}$.

In general, we work with the two pairs of elements γ_H, γ_G and $\overline{\gamma}_H, \overline{\gamma}_G$. There exist $h, \overline{h} \in G^*_{\mathrm{sc}}$ such that

$$h\psi(\gamma_G)h^{-1} = \gamma \text{ and } \overline{h}\psi(\overline{\gamma}_G)\overline{h}^{-1} = \overline{\gamma}.$$

Set

$$v(\sigma) = hu(\sigma)\sigma(h)^{-1} \text{ and } \overline{v}(\sigma) = \overline{h}u(\sigma)\sigma(\overline{h})^{-1},$$

where $u(\sigma) \in G^*_{\mathrm{sc}}$ and $\psi\sigma(\psi)^{-1} = \mathrm{Int}\, u(\sigma), \sigma \in \Gamma$. Then $v(\sigma)$ and $\overline{v}(\sigma)$ are cochains of Γ in T_{sc} and $\overline{T}_{\mathrm{sc}}$, each well defined up to coboundaries because γ and $\overline{\gamma}$ are strongly regular.

Further, $\partial v = \partial \overline{v} = \partial u$, each coboundary taking values in the center Z_{sc} of G^*_{sc}. Let $U = U(T, \overline{T})$ be the torus

$$T_{\mathrm{sc}} \times \overline{T}_{\mathrm{sc}}/\{(z^{-1}, z) : z \in Z_{\mathrm{sc}}\}.$$

Then $\sigma \to (v(\sigma)^{-1}, \overline{v}(\sigma))$ defines an element of $H^1(U)$ which is independent of the choices for $u(\sigma), h$ and $\overline{h}$. We write this class as

$$\mathrm{inv}\left(\frac{\gamma_H, \gamma_G}{\overline{\gamma}_H, \overline{\gamma}_G}\right).$$

From $T_{\mathrm{sc}} \times \overline{T}_{\mathrm{sc}} \to U$ we get $H^1(T_{\mathrm{sc}}) \times H^1(\overline{T}_{\mathrm{sc}}) \to H^1(U)$. If G is quasi-split then $\mathrm{inv}\left(\frac{\gamma_H, \gamma_G}{\overline{\gamma}_H, \overline{\gamma}_G}\right)$ is the image of $(\mathrm{inv}(\gamma_H, \gamma_G)^{-1}, \mathrm{inv}(\overline{\gamma}_H, \overline{\gamma}_G))$.

Recall that $\widehat{T}_{\mathrm{ad}}$ is $\widehat{T}/Z(\widehat{G})$, the torus dual to the preimage T_{sc} of T in G_{sc}. We denote by $\widehat{T}_{\mathrm{sc}}$ the torus dual to $T_{\mathrm{ad}} = T/Z(G)$. Then the center $\widehat{Z}_{\mathrm{sc}}$ of the simply-connected covering of the derived group of $\widehat{G}$, which is a finite group isomorphic to Z_{sc}, is canonically embedded in $\widehat{T}_{\mathrm{sc}}$ and $\widehat{\overline{T}}_{\mathrm{sc}}$. Set

$$\widehat{U} = \widehat{T}_{\mathrm{sc}} \times \widehat{\overline{T}}_{\mathrm{sc}}/\{z, z) : z \in Z_{\mathrm{sc}}\}.$$

Then $X^*(\widehat{U}) \subset X^*(\widehat{T}_{\mathrm{sc}} \times \widehat{\overline{T}}_{\mathrm{sc}})$. At the same time, $X^*(U) \subset X^*(T_{\mathrm{sc}} \times \overline{T}_{\mathrm{sc}})$. The **Q**-pairing between $X^*(\widehat{T}_{\mathrm{sc}} \times \widehat{\overline{T}}_{\mathrm{sc}})$ and $X^*(T_{\mathrm{sc}} \times \overline{T}_{\mathrm{sc}})$ yields a dual Z-pairing between $X^*(\widehat{U})$ and $X^*(U)$, and so $\widehat{U}$ is the torus dual to U.

To the endoscopic datum s we attach $\mathbf{s}_U \in \pi_0(\widehat{U}^\Gamma)$ as follows. Suppose $\tilde{s}$ lies in the preimage under $\mathcal{T}_{\mathrm{sc}} \to \mathcal{T}_{\mathrm{ad}}$ of the projection of s onto $\mathcal{T}_{\mathrm{ad}} = \mathcal{T}/Z(\widehat{G})$. From $\mathcal{T} \to \widehat{T}$ and $\mathcal{T} \to \widehat{\overline{T}}$ we obtain $\mathcal{T}_{\mathrm{sc}} \to \widehat{T}_{\mathrm{sc}}$ and $\mathcal{T}_{\mathrm{sc}} \to \widehat{\overline{T}}_{\mathrm{sc}}$. The images $\tilde{s}_T$ and $\tilde{s}^{\overline{T}}$ of $\tilde{s}$ depend only on the embeddings $T_H \to T$ and $\overline{T}_H \to \overline{T}$. The image s_U of $(\tilde{s}_T, \tilde{s}^{\overline{T}})$ in $\widehat{U}$ is independent of the choice of $\tilde{s}$. It is also Γ-invariant and so defines an element $\mathbf{s}_U$ of $\pi_0(\widehat{U}^\Gamma)$.

Definition $\Delta_1(\gamma_H, \gamma_G; \overline{\gamma}_H, \overline{\gamma}_G) = \left\langle \mathrm{inv}\left(\frac{\gamma_H, \gamma_G}{\overline{\gamma}_H, \overline{\gamma}_G}\right), \mathbf{s}_U \right\rangle$.

Note that if G is quasi-split over F then

$$\Delta_1(\gamma_H, \gamma_G; \overline{\gamma}_H, \overline{\gamma}_G) = \langle \mathrm{inv}(\gamma_H, \gamma_G), \mathbf{s}_T\rangle^{-1}\langle \mathrm{inv}(\overline{\gamma}_H, \overline{\gamma}_G), \mathbf{s}_{\overline{T}}\rangle.$$

Lemma 3.4.A. *If $T_H \to T$ and $\overline{T}_H \to \overline{T}$ are replaced by their g- and $\overline{g}$-conjugates, where $g \in \mathfrak{A}(T_{\rm sc})$ and $\overline{g} \in \mathfrak{A}(\overline{T}_{\rm sc})$, then $\Delta_1(\gamma_H, \gamma_G; \overline{\gamma}_H, \overline{\gamma}_G)$ is multiplied by*

$$\langle \mathbf{g}_T, \mathbf{s}_T \rangle \langle \mathbf{g}_{\overline{T}}, \mathbf{s}_{\overline{T}} \rangle^{-1},$$

where $\mathbf{g}_T$ is the class of $\sigma \to g\sigma(g^{-1})$ in $H^1(T_{\rm sc})$, and $\mathbf{g}_{\overline{T}}$ the class of $\sigma \to \overline{g}\sigma(\overline{g}^{-1})$ in $H^1(\overline{T}_{\rm sc})$.

Proof. $v(\sigma)$ is replaced by $g^{-1}v(\sigma)\sigma(g)$ and $g(g^{-1}v(\sigma)\sigma(g))g^{-1} = v(\sigma) \cdot (g\sigma(g)^{-1})^{-1}$. Similarly for $\overline{v}(\sigma)$ and the lemma follows.

(3.5) *Term Δ_{III_2} or Δ_2*

For the construction here we fix Borel subgroups $B_H \supset T_H, B \supset T$ which yield the isomorphism $\widehat{T}_H \to \mathcal{T}_H \to \mathcal{T} \to \widehat{T}$ dual to $T_H \to T$. To the χ-data $\{\chi_\alpha\}$ are attached admissible embeddings $\xi_T : {}^LT \to {}^LG$ extending $\widehat{T} \to \mathcal{T}$ and $\xi_{T_H} : {}^LT_H \to {}^LH$ extending $\widehat{T}_H \to \mathcal{T}_H$. Then

$$\xi \cdot \xi_{T_H} = a\xi_T,$$

where a is a 1-cocycle of W_F in $\mathcal{T}$ for the transport of the action of W_F on $\widehat{T}$. We transport a to T without change in notation. Its class **a** in $H^1(W_F, \widehat{T})$ is independent of the choice of B_H and B by (2.6.2), and of the Γ-splittings $(\mathcal{B}_H, \mathcal{T}_H, \{X^H\})$ and $(\mathcal{B}, \mathcal{T}, \{X\})$ by (2.6.1). Further if $(T_H \to T, \{\chi_\alpha\})$ is replaced by its g-conjugate, $g \in \mathfrak{A}(T)$, then **a** is replaced by its image in $H^1(W_F, \widehat{T}^g)$ under the map induced by $\operatorname{Int} g^{-1}$ [see (2.6.4)].

Definition. $\Delta_2(\gamma_H, \gamma_G) = \langle \mathbf{a}, \gamma \rangle$.

Clearly, replacing $(T_H \to T, \{a_\alpha\}, \{\chi_\alpha\})$ by an $\mathfrak{A}(T)$-conjugate has no effect on $\Delta_2(\gamma_H, \gamma_G)$.

Lemma 3.5.A. *Suppose that the χ-data $\{\chi_\alpha\}$ are replaced by $\{\chi'_\alpha\}$, where $\chi'_\alpha = \chi_\alpha \zeta_\alpha$. Then $\Delta_2(\gamma_H, \gamma_G)$ is multiplied by*

$$\prod^{\rm asymm} \zeta_\alpha(\gamma^\alpha)^{-1} \prod^{\rm symm} \zeta_\alpha(\delta^\alpha)^{-1},$$

where the product $\prod^{\rm asymm}$ is over representatives α for the pairs $\pm\mathcal{O}$ of asymmetric orbits outside H and $\prod^{\rm symm}$ is over representatives α for the symmetric orbits outside H.

The elements γ^α and δ^α were defined in (3.3), and will be recalled in the proof.

Proof. According to (2.6.3) **a** is replaced by $\mathbf{ac}^{-1}$ where **c** is represented by the cocycle

$$c(w) = \prod_\alpha \prod_{i=1}^n \zeta_\alpha(v_0(u_i(w)))^{\sigma_i^{-1}\alpha}.$$

Here the product $\prod_\alpha$ is over representatives α for pairs $\pm\mathcal{O}$ of Γ-orbits of roots of T that lie outside H. The elements $v_0(u_i(w)), \sigma_i^{-1}\alpha$ were defined in (2.5). Note also Corollary 2.5.B.

Suppose that $\mathcal{O}$ is a symmetric orbit outside H. As in (3.3) we have

$$T(F) \to T^{\mathcal{O}}(F) \cong T^{\alpha}(F_{\alpha}).$$

Then $\gamma^{\mathcal{O}}$ denotes the image of γ in $T^{\mathcal{O}}(F)$ and γ^{α} its image in $T^{\alpha}(F_{\alpha})$; $\gamma^{\alpha} = \delta^{\alpha}\overline{\delta^{\alpha}}$. We have also

$$H^1(W_{F_{\pm\alpha}}, \widehat{T^{\alpha}}) \cong H^1(W_F, \widehat{T^{\mathcal{O}}}) \to H^1(W_F, \widehat{T}).$$

We pull $\mathbf{c}$ back to $\mathbf{c}_{\mathcal{O}}$ in $H^1(W_F, \widehat{T^{\mathcal{O}}})$ and let $\mathbf{c}_{\alpha}$ be the corresponding element of $H^1(W_{F_{\pm\alpha}}, \widehat{T^{\alpha}})$. Then

$$\langle \mathbf{c}, \gamma \rangle = \langle \mathbf{c}_{\mathcal{O}}, \gamma^{\mathcal{O}} \rangle$$

by the functoriality of $\langle , \rangle$ which follows from its definition [B, Sect. 9]. Moreover

$$\langle \mathbf{c}_{\mathcal{O}}, \gamma^{\mathcal{O}} \rangle = \langle \mathbf{c}_{\alpha}, \gamma^{\alpha} \rangle$$

because the pairing $\langle , \rangle$ respects restriction of scalars (see [B]). To compute $\langle \mathbf{c}_{\alpha}, \gamma^{\alpha} \rangle$ we introduce S_{α}, the group obtained from G_m by restriction of scalars from F_{α} to $F_{\pm\alpha}$, and the obvious homomorphism $S_{\alpha} \to T_{\alpha}$, surjective on $F_{\pm\alpha}$-valued points. By functoriality, we may replace $\mathbf{c}_{\alpha}$ by its image in $H^1(W_{\pm\alpha}, \widehat{S}_{\alpha})$ and conclude that

$$\langle \mathbf{c}_{\alpha}, \gamma^{\alpha} \rangle = \zeta_{\alpha}(\alpha(\delta^{\alpha})).$$

For asymmetric $\mathcal{O}$ we have

$$T(F) \to T^{\pm\mathcal{O}}(F) \cong T^{\alpha}(F_{\alpha})$$

as in (3.3), and dual

$$H^1(W_{F_{\alpha}}, \widehat{T^{\alpha}}) \cong H^1(W_F, \widehat{T^{\pm\mathcal{O}}}) \to H^1(W_F, \widehat{T}).$$

again it is sufficient to compute $\langle \mathbf{c}_{\alpha}, \gamma^{\alpha} \rangle$. Now T^{α} is split over F_{α} and from $\mathbf{c}_{\alpha}(w) = \zeta_{\alpha}(w)^{\alpha}, w \in W_{F_{\alpha}}$, we obtain

$$\langle \mathbf{c}_{\alpha}, \gamma^{\alpha} \rangle = \zeta_{\alpha}(\alpha(\gamma^{\alpha})).$$

This completes the proof of the lemma.

(3.6) *Term* Δ_{IV}

If $\gamma \in T(F)$ then $\prod_{\alpha}(\alpha(\gamma) - 1)$, where the product is over *all* roots of T in G^*, lies in F. We set

$$D_{G^*}(\gamma) = |\prod_{\alpha}(\alpha(\gamma) - 1)|^{1/2}.$$

Definition. $\Delta_{IV}(\gamma_H, \gamma_G) = D_{G^*}(\gamma) D_H(\gamma_H)^{-1}$.

Then $\Delta_{IV}(\gamma_H, \gamma_G)$ depends only on the stable conjugacy classe of γ_H.

(3.7) *The Factor* Δ

We now fix the pair $\overline{\gamma}_H, \overline{\gamma}_G$ and specify $\Delta(\overline{\gamma}_H, \overline{\gamma}_G)$ arbitrarily. Then we define

$$\Delta(\gamma_H, \gamma_G) = \Delta(\overline{\gamma}_H, \overline{\gamma}_G)\Delta(\gamma_H, \gamma_G; \overline{\gamma}_H, \overline{\gamma}_G),$$

where

$$\Delta(\gamma_H, \gamma_G; \overline{\gamma}_H, \overline{\gamma}_G)$$

is equal to

$$\frac{\Delta_I(\gamma_H, \gamma_G)}{\Delta(\overline{\gamma}_H, \overline{\gamma}_G)} \cdot \frac{\Delta_{II}(\gamma_H, \gamma_G)}{\Delta_{II}(\overline{\gamma}_H, \overline{\gamma}_G)} \cdot \frac{\Delta_{III_2}(\gamma_H, \gamma_G)}{\Delta_{III_2}(\overline{\gamma}_H, \overline{\gamma}_G)} \cdot \frac{\Delta_{IV}(\gamma_H, \gamma_G)}{\Delta_{IV}(\overline{\gamma}_H, \overline{\gamma}_G)} \cdot \Delta_{III_1}(\gamma_H, \gamma_G; \overline{\gamma}_H, \overline{\gamma}_G).$$

In the case that G is quasi-split over F we set

$$\Delta_0(\gamma_H, \gamma_G) = \Delta_I(\gamma_H, \gamma_G)\Delta_{II}(\gamma_H, \gamma_G)\Delta_1(\gamma_H, \gamma_G)\Delta_2(\gamma_H, \gamma_G)\Delta_{IV}(\gamma_H, \gamma_G)$$

so that

$$\Delta(\gamma_H, \gamma_G; \overline{\gamma}_H, \overline{\gamma}_G) = \Delta_0(\gamma_H, \gamma_G)/\Delta_0(\overline{\gamma}_H, \overline{\gamma}_G).$$

Recall that $\Delta_I(\gamma_H, \gamma_G)$, *but not* $\Delta_I(\gamma_H, \gamma_G)/\Delta_I(\overline{\gamma}_H, \overline{\gamma}_G)$, depends on the choice of an F-splitting for G^*.

Theorem 3.7.A. $\Delta(\gamma_H, \gamma_G)$ *is independent of the choice of admissible embeddings, a-data and χ-data.*

Proof. If $T_H \to T, \overline{T}_H \to \overline{T}$ and their a-data, χ-data are replaced by $\mathfrak{A}(T)$-, $\mathfrak{A}(\overline{T})$-conjugates then only Δ_I and Δ_1 are changed. By Lemmas 3.2.B and 3.4.A, Δ is unchanged. If the a-data and χ-data alone are changed then Δ_I, Δ_{II}, and Δ_2 are affected. Again the effects cancel, by Lemmas 3.2.C, 3.3.C, 3.3.D, and 3.5.A.

The same lemmas show that the factor $\Delta_0(\gamma_H, \gamma_G)$ is independent of these choices.

Finally, if no strongly regular element in $G(F)$ has an image in $H(F)$ we set $\Delta \equiv 0$.

4. Some Properties of Δ

(4.1) *Invariance*

Lemma 4.1.A. $\Delta(\gamma_H^1, \gamma_G^1, \gamma_H^2, \gamma_G^2)\Delta(\gamma_H^2, \gamma_G^2; \gamma_H^3, \gamma_G^3) = \Delta(\gamma_H^1, \gamma_G^1; \gamma_H^3, \gamma_G^3)$.

Proof. It is enough to show this with Δ replaced by $\Delta_1 = \Delta_{III_1}$, that is to show that

$$\langle \mathrm{inv}_{(1,2)}, \mathbf{s}_{U_{1,2}}\rangle\langle \mathrm{inv}_{(2,3)}, \mathbf{s}_{U_{2,3}}\rangle = \langle \mathrm{inv}_{(1,3)}, \mathbf{s}_{U_{1,3}}\rangle,$$

where

$$\mathrm{inv}_{(i,j)} = \mathrm{inv}\left(\frac{\gamma_H^i, \gamma_G^i}{\gamma_H^j, \gamma_G^j}\right), \quad U_{i,j} = U(T^i, T^j),$$

$1 \leqq i < j \leqq 3$, and T^i is the image of Cent(γ_H^i, H) under some admissible embedding in G^*. Let

$$V = T_{\mathrm{sc}}^1 \times T_{\mathrm{sc}}^2 \times T_{\mathrm{sc}}^3/\{(z^{-1}, zw^{-1}, w) : z, w \in Z_{\mathrm{sc}}\}$$

in the notation of (3.4). There are F-homomorphisms $U_{i,j} \to V$. Under the induced maps on cohomology the image of $\mathrm{inv}_{(1,3)}$ is the product of the images of $\mathrm{inv}_{(1,2)}$ and $\mathrm{inv}_{(2,3)}$. Still following the notation of (3.4) we see that the dual of V is

$$\widehat{V} = \widehat{T}_{\mathrm{sc}}^1 \times \widehat{T}_{\mathrm{sc}}^2 \times \widehat{T}_{\mathrm{sc}}^3/\{(z, z, z) : z \in \widehat{Z}_{\mathrm{sc}}\}.$$

Let $\mathbf{s}_V$ be the image of $(\tilde{\mathbf{s}}_{T^1}, \tilde{\mathbf{s}}_{T^2}, \tilde{\mathbf{s}}_{T^3})$ in $\pi_0(\widehat{V}^\Gamma)$. Then

$$\langle \mathrm{inv}_{(1,2)}, \mathbf{s}_{U_{1,2}}\rangle\langle \mathrm{inv}_{(2,3)}, \mathbf{s}_{U_{2,3}}\rangle = \langle \text{image}(\mathrm{inv}_{(1,2)})\text{image}(\mathrm{inv}_{(2,3)}), \mathbf{s}_V\rangle$$

which equals

$$\langle \text{image}(\mathrm{inv}_{(1,3)}), \mathbf{s}_V\rangle = \langle \mathrm{inv}_{(1,3)}, \mathbf{s}_{U_{1,3}}\rangle$$

and the lemma is proved.

Corollary 4.1.B.

(i) $\Delta(\gamma_H, \gamma_G; \gamma_H, \gamma_G) = 1$ *and*

(ii) $\Delta(\gamma'_H, \gamma'_G) = \Delta(\gamma_H, \gamma_G)\Delta(\gamma'_H, \gamma'_G; \gamma_H, \gamma_G)$ if γ_H, γ'_H *are images of* γ_G, γ'_G *respectively.*

Lemma 4.1.C. $\Delta(\gamma_H, \gamma_G)$ *depends only on the stable conjugacy class of* γ_H *in* $H(F)$ *and the conjugacy class of* γ_G *in* $G(F)$.

Proof. Let $\gamma'_G = g^{-1}\gamma_G g, g \in G(F)$. Then

$$\Delta(\gamma_H, \gamma'_G) = \Delta(\gamma_H, \gamma_G)\Delta(\gamma_H, \gamma'_G; \gamma_H, \gamma_G).$$

On examining the terms $\Delta_I, \ldots, \Delta_{IV}$ we see that

$$\Delta(\gamma_H, \gamma'_G; \gamma_H, \gamma_G) = \Delta_1(\gamma_H, \gamma'_G; \gamma_H, \gamma_G)$$

and so it remains to check that $\Delta_1(\gamma_H, \gamma'_G; \gamma_H, \gamma_G) = 1$. There is $g_1 \in G_{\mathrm{sc}}(\overline{F})$ such that $g^{-1}\gamma_G g = g_1^{-1}\gamma_G g_1$. Then $g_1\sigma(g_1)^{-1}$ is a cocycle with values in $\mathrm{Ker}(G_{\mathrm{sc}} \to G)$. If $h \in G^*_{\mathrm{sc}}$ is such that $h\psi(\gamma_G)h^{-1} = \gamma$ then $h\psi(g_1)\psi(\gamma'_G)(h\psi(g_1))^{-1} = \gamma$ and so the cocycle defining inv $\left(\frac{\gamma_H, \gamma'_G}{\gamma_H, \gamma_G}\right)$ is of the form $(\psi(g_1\sigma(g_1)^{-1})^{-1}v(\sigma)^{-1}, v(\sigma))$. Then $\Delta_1(\gamma_H, \gamma'_G; \gamma_H, \gamma_G)$ coincides with $\langle\psi(g_1\sigma(g_1)^{-1})^{-1}, \mathbf{s}_T\rangle \cdot \Delta_1(\gamma_H, \gamma_G; \gamma_H, \gamma_G)$. The first term in this product is trivial since $\psi(g_1\sigma(g_1)^{-1}) \in \mathrm{Ker}(G^*_{\mathrm{sc}} \to G^*)$; the second is trivial by (4.1.B).

Suppose $\gamma'_H = h^{-1}\gamma_H h$ is stably conjugate to γ_H. Then an admissible embedding of Cent(γ'_H, H) in G^* is obtained by composition of an admissible embedding of Cent(γ_H, H) with $\mathrm{Int}\ h$. Term-by-term examination of Δ yields $\Delta(\gamma'_H, \gamma_G; \gamma_H, \gamma_G) = 1$.

(4.2) *The Local Hypothesis*

We now examine the relation between factors for G and those for G^*.

The endoscopic data $(H, \mathcal{H}, s, \xi)$ serve both G and G^*. Suppose that strongly G-regular $\gamma_H \in H(F)$ is an image of $\gamma_G \in G(F)$. Then γ_H is also strongly G^*-regular. By Steinberg's Theorem [K1], γ_H is the image of a stable conjuacy class of elements in $G^*(F)$. Suppose that γ_{G^*} belongs to this class. Then both $\Delta(\gamma_H, \gamma_G)$ and $\Delta(\gamma_H, \gamma_{G^*})$ are defined and nonzero. Set

$$\Delta_{G/G^*}(\gamma_H, \gamma_G, \gamma_{G^*}) = \Delta(\gamma_H, \gamma_G)/\Delta(\gamma_H, \gamma_{G^*}).$$

It is clear from the definitions that this assumes a quite simple form, for we may use the same auxiliary data of admissible embeddings, a-data and χ-data to define all terms in the numerator and denominator. Since $\Delta(\gamma_H, \gamma_G)$ and $\Delta(\gamma_H, \gamma_{G^*})$ are canonical only up to constants we will investigate

$$\Delta_{G/G^*}(\gamma_H, \gamma_G, \gamma_{G^*})/\Delta_{G/G^*}(\gamma'_H, \gamma'_G, \gamma'_{G^*})$$

where γ'_H is an image of γ'_G and of γ'_{G^*}.

First we shall define a number $\lambda_H(\gamma_G, \gamma_{G^*}, \gamma'_G, \gamma_{G^*})$. There are unique admissible embeddings Cent$(\gamma_H, H) \rightarrow$ Cent(γ_{G^*}, G^*) and Cent$(\gamma'_H, H) \rightarrow$ Cent(γ'_{G^*}, G^*) mapping γ_H to γ_{G^*} and γ'_H to γ'_{G^*}. We set

$$\lambda_H(\gamma_G, \gamma_{G^*}; \gamma'_G, \gamma'_{G^*}) = \left\langle \operatorname{inv}\left(\frac{\gamma_H, \gamma_G}{\gamma'_H, \gamma'_G}\right), \mathbf{s}_U \right\rangle$$

in the notation of (3.4). Note that $\operatorname{inv}\left(\frac{\gamma_H, \gamma_G}{\gamma'_H, \gamma'_G}\right)$ is represented by the cocycle

$$(\sigma(h)u(\sigma)^{-1}h^{-1}, h'u(\sigma)\sigma(h')^{-1}),$$

where $h, h' \in G^*_{\mathrm{sc}}$ and

$$h\psi(\gamma_G)h^{-1} = \gamma_{G^*}, \quad h'\psi(\gamma'_G){h'}^{-1} = \gamma'_{G^*}.$$

Moreover $u(\sigma) \in G^*_{\mathrm{sc}}$ is given by $\psi\sigma(\psi)^{-1} = \operatorname{Int} u(\sigma)$. Then λ_H is independent of the choice of $\gamma_G, \ldots, \gamma'_{G^*}$ within their conjugacy classes (see the proof of Lemma 4.1.C).

Lemma 4.2.A. $\Delta_{G/G^*}(\gamma_H, \gamma_G, \gamma_{G^*})/\Delta_{G/G^*}(\gamma'_H, \gamma'_G, \gamma'_{G^*}) = \lambda_H(\gamma_G, \gamma_{G^*}; \gamma'_G, \gamma'_{G^*})$.

Proof. We choose admissible embeddings as in the definition of λ_H. Since $\Delta_I, \Delta_{II}, \Delta_2, \Delta_{IV}$ take the same values at (γ_H, γ_G) as at (γ_H, γ_{G^*}) only Δ_1 yields a nontrivial contribution to the left side. In view of Corollary 4.1.B this contribution is

$$\Delta_1(\gamma_H, \gamma_G; \gamma'_H, \gamma'_G)/\Delta_1(\gamma_H, \gamma_{G^*}; \gamma'_H, \gamma'_{G^*}) = \Delta_1(\gamma_H, \gamma_G; \gamma'_H, \gamma'_G)$$

by our choice of embeddings. This equals

$$\lambda_H(\gamma_G, \gamma_{G^*}; \gamma'_G, \gamma'_H),$$

and the lemma is proved.

The following is then immediate:

Corollary 4.2.B.

$$\frac{\Delta(\gamma_H, \gamma_G)}{\Delta(\gamma_H, \gamma_{G^*})} = \lambda_H(\gamma_G, \gamma_{G^*}; \gamma'_G, \gamma'_{G^*}) \frac{\Delta(\gamma'_H, \gamma'_G)}{\Delta(\gamma'_H, \gamma'_{G^*})}.$$

This asserts that the factors Δ satisfy the Local Hypothesis of [L2, Chap. VI]. To reconcile our notation with that of [L2] we note that if $T_H = \text{Cent}(\gamma_H, H), T = \text{Cent}(\gamma_{G^*}, G^*)$ and $T_G = \text{Cent}(\gamma_G, G)$ then

$$\begin{array}{ccc} T_H \to \mathbf{T}_H \to \mathbf{T} \to & T \\ & \uparrow \\ & T_G \end{array}$$

is a *diagram* D, where $T_H \to T$ is the admissible embedding taking γ_H to γ_{G^*} and $\operatorname{Int} h \circ \psi : T_G \to T$. Similarly we have D':

$$\begin{array}{ccc} T'_H \to \mathbf{T}_H \to \mathbf{T} \to & T' \\ & \uparrow \\ & T'_G. \end{array}$$

The Local Hypothesis states that $c(D, D')$, the left side of Lemma 4.2.A, is given by an expression $\kappa(\theta(E, E'))$ which we now show to coincide with $\lambda_H = \lambda_H(\gamma_G, \gamma_{G^*}, \gamma'_G, \gamma'_{G^*})$.

There is no harm in assuming G simply-connected. Let $U = U(T, T')$. Then we have

$$X_*(U) \hookrightarrow X_*(T_{\text{ad}}) \times X_*(T'_{\text{ad}}).$$

The elements $\lambda \in X_*(T_{\text{ad}}), \lambda' \in X_*(T'_{\text{ad}})$ are defined on p. 84 of [L2]. On modifying λ' as on p. 85 we may assume (6.12) of p. 85. Then $\theta(E, E') = \lambda - \lambda'$. On the other hand, $(\lambda, \lambda') \in X_*(U)$ and defines an element of $H^{-1}(X_*(U))$ which corresponds under Tate-Nakayama duality to the class of the cocycle

$$(\sigma(h)u(\sigma)^{-1}h^{-1}, h'u(\sigma)\sigma(h')^{-1})$$

defining $\operatorname{inv}\left(\frac{\gamma_H, \gamma_G}{\gamma'_H, \gamma'_G}\right)$ in $H^1(U)$. Since κ is obtained from the endoscopic datum s [L2, p. 100] we conclude that $\kappa(\theta(E, E'))$ coincides with λ_H.

(4.3) *Extension to All G-Regular Elements*

The definition of $\Delta_1(\gamma_H, \gamma_G; \overline{\gamma}_H, \overline{\gamma}_G)$ requires that G-regular γ_H be strongly regular. The notion of image, however, is well defined for an arbitrary G-regular semisimple element in $H(F)$ [recall (1.3)] and we expect an identity

$$\Phi^{\text{st}}(\gamma_H, f^H) = \sum_{\gamma_G} \Delta(\gamma_H, \gamma_G)\Phi(\gamma_G, f^G)$$

for all G-regular semisimple elements γ_H in $H(F)$, where $\Delta(\gamma_H, \gamma_G) = 0$ unless γ_H is an image of γ_G.

We shall extend Δ by continuity. Suppose $\gamma_H^0 \in T_H(F)$ is G-regular and is an image of $\gamma_G^0 \in T_G(F)$. Fix an admissible embedding $T_H \to T$ of T_H in G^* and an isomorphism $\operatorname{Int} x \circ \psi : T \to T_G$ over F so that $\gamma_H^0 \to \gamma_G^0$ under $T_H \to T_G$. Suppose $g_1, \ldots, g_n$ are representatives for $\mathcal{D}(T_G) = T_G(\overline{F}) \backslash \mathfrak{A}(T_G)/G(F)$, where $\mathfrak{A}(T_G) = \{g \in G(\overline{F}) : g\sigma(g)^{-1} \in T_G(\overline{F})\}$. Then if $\gamma_G \in T_G(F)$ is strongly regular the elements $g_i^{-1}\gamma_G g_i$ are representatives for the conjugacy classes in the stable conjugacy class of γ_G. We may find a sequence $\{\gamma_H\}$ of strongly G-regular elements in $T_H(F)$ such that $\{\gamma_H\} \to \gamma_H^0$. Let γ_G be the image of γ_H under $T_H \to T_G$. Then the limit $\gamma_G^i = g_i^{-1}\gamma_G^0 g_i$ of $\{g_i^{-1}\gamma_G g_i\}$ has γ_H^0 as image. We define

$$\Delta(\gamma_H^0, \gamma_G^i) = \lim_{\gamma_H \to \gamma_H^0} \Delta(\gamma_H, g_i^{-1}\gamma_G g_i),$$

as an examination of the terms in Δ shows the right side to be well-defined. Then if f, f^H have Δ-matching orbital integrals we have

$$\Phi^{\mathrm{st}}(\gamma_H^0, f^H) = \sum_{i=1}^{n} \Delta(\gamma_H^0, \gamma_G^i)\Phi(\gamma_G^i, f),$$

where $\Phi(\gamma_G^i, f)$ is as specified in (1.3) and

$$\Phi^{\mathrm{st}}(\gamma_H^0, f^H) = \sum_{j=1}^{m} \Phi(\gamma_H^j, f^H),$$

with $\gamma_H^j = h_j^{-1}\gamma_H^0 h_j$ and $\{h_j\}$ representatives for $\mathcal{D}(T_H)$. Thus on either side a conjugacy class may contribute several terms.

(4.4) *Passage to Central Extensions*

The center of G is canonically embedded as a central subgroup of H. If γ_H is an image of γ_G then $z\gamma_H$ is an image of $z\gamma_G$, $z \in Z(F)$. Still assuming $\mathcal{H}$ is ${}^L H$ we have:

Lemma 4.4.A. *There is a character λ^G on $Z(F)$ such that*

$$\Delta(z\gamma_H, z\gamma_G) = \lambda^G(z)\Delta(\gamma_H, \gamma_G), \quad z \in Z(F),$$

for all γ_H, γ_G.

The proof will be included in another paper. There is one case which it is useful to treat here.

Proof for z in the identity component Z^0 of Z:

According to the definitions,

$$\Delta(z\gamma_H, z\gamma_G)\Delta(\gamma_H, \gamma_G)^{-1} = \Delta_2(z\gamma_H, z\gamma_G)\Delta_2(\gamma_H, \gamma_G)^{-1} = \langle \mathbf{a}, z \rangle,$$

so that we have to show that the character $z \to \langle \mathbf{a}, z\rangle$ on $Z^0(F)$ is independent of the choice of γ_H and of $T_H \to T$, χ-data and a-data. The last three choices have no effect because they have no effect on Δ.

Given also $\overline{\gamma}_H$ and $\overline{T}_H \to \overline{T}$ we form $S = T \times \overline{T}/Z^0, S_H = T_H \times \overline{T}_H/Z^0, \widetilde{G} = G \times G/Z^0$ and $\widetilde{H} = H \times H/Z^0$, where Z^0 is embedded in each case by $z \to (z, z^{-1})$. Because Z^0 is connected we have natural embeddings

$$ {}^L S \hookrightarrow {}^L T \times {}^L \overline{T}, \quad {}^L S_H \hookrightarrow {}^L T_H \times {}^L \overline{T}_H, \quad {}^L \widetilde{G} \hookrightarrow {}^L G \times {}^L G, \quad {}^L \widetilde{H} \hookrightarrow {}^L H \times {}^L H $$

and may form commutative diagrams

$$\begin{tikzcd} {}^L\widetilde{H} \arrow[r] \arrow[d, hook, "\widetilde{\xi}"'] & {}^L H \times {}^L H \arrow[d, hook, "(\xi,\xi)"] \\ {}^L\widetilde{G} \arrow[r] & {}^L G \times {}^L G \end{tikzcd} \qquad \begin{tikzcd} {}^L S \arrow[r] \arrow[d, hook, "\xi_S"'] & {}^L T \times {}^L \overline{T} \arrow[d, hook, "(\xi_T, \xi_{\overline{T}})"] \\ {}^L\widetilde{G} \arrow[r] & {}^L G \end{tikzcd}$$

and so on. As a result we conclude that the element $(\mathbf{a}, \overline{\mathbf{a}})$ of $H^1(W_F, \widehat{T} \times \widehat{T})$ is the image of an element $\tilde{\mathbf{a}}$ of $H^1(W_F, \widehat{S})$. Then

$$ \langle \mathbf{a}, z\rangle \langle \overline{\mathbf{a}}, z\rangle^{-1} = \langle (\mathbf{a}, \overline{\mathbf{a}}), (z, z^{-1})\rangle = \langle \tilde{\mathbf{a}}, 1\rangle = 1 $$

and the lemma is proved.

We now remove the assumption that $\mathcal{H}$ is an L-group. The data $(H, \mathcal{H}, s, \xi)$ given, orbital integrals of functions on $G(F)$ will be matched not with those of functions on $H(F)$ but with those of functions on $H_1(F)$, where H_1 is a central extension of H. We shall take H_1 attached to a central extension of G as the arguments are more transparent. Thus we fix a z-extension $1 \to Z_1 \to G_1 \to G \to 1$ of G [K1]. This means, in particular, that Z_1 is a connected central subgroup of $G_1, G_1(F) \to G(F)$ is surjective and the derived group of G_1 is simply-connected. The dual sequence $1 \to \widehat{G} \to \widehat{G}_1 \to \widehat{Z}_1 \to 1$ allows us to regard $\widehat{G}$ as a subgroup of $\widehat{G}_1$. We may assume that $\varrho_{G_1}(\sigma)$ and $\varrho_G(\sigma)$ agree on $\widehat{G}, \sigma \in \Gamma$, so that ${}^L G$ embeds canonically in ${}^L G_1$.

There is a central extension $1 \to Z_1 \to H_1 \to H \to 1$ and embedding $\xi_1 : {}^L H_1 \hookrightarrow {}^L G_1$ such that $(H_1, {}^L H_1, s, \xi_1)$ are endoscopic data for G_1 (see [L1]). The parameter for a character λ on $Z_1(F)$ is given by

$$ W_F \to {}^L G_1 \to {}^L Z_1, $$

where the first arrow denotes the restriction of ξ_1 to W_F and the second is the natural extension of $\widehat{G}_1 \to \widehat{Z}_1$.

The matching is to be between orbital integrals of functions f on $G(F)$, and thus of functions on $G_1(F)$ invariant under $Z_1(F)$, and orbital integrals of functions f^{H_1} on $H_1(F)$ satisfying

$$ f^{H_1}(zh) = \lambda(z) f^{H_1}(h), \quad z \in Z_1(F), \quad h \in H_1(F). $$

We need consider only elements γ_{H_1} whose image in $H(F)$ is strongly G-regular. Then γ_{H_1} is *an image* of $\gamma_G \in G(F)$ if it is an image of some γ_{G_1} in the preimage of γ_G in $G_1(F)$. The element γ_{G_1} is uniquely determined. We say that f and f^{H_1} have Δ-matching orbital integrals if, as usual,

$$\Phi^{\mathrm{st}}(\gamma_{H_1}, f^{H_1}) = \sum_{\gamma_G} \Delta(\gamma_{H_1}, \gamma_G)\Phi(\gamma_G, f)$$

for all such γ_{H_1}. Because of the transformation rule for f^{H_1} we must have

$$\Delta(z\gamma_{H_1}, \gamma_G) = \lambda(z)\Delta(\gamma_{H_1}, \gamma_G), \quad z \in Z_1(F).$$

The factor $\Delta(\gamma_{H_1}, \gamma_{G_1})$ has been defined. We set

$$\Delta(\gamma_{H_1}, \gamma_G) = \Delta(\gamma_{H_1}, \gamma_{G_1})$$

if γ_{H_1} is an image of γ_{G_1} and γ_G is the image of γ_{G_1} under $G_1(F) \to G(F)$, or $\Delta(\gamma_{H_1}, \gamma_G) = 0$ if γ_{H_1} is not an image of γ_G.

Recall that

$$\Delta(z\gamma_{H_1}, zy_{G_1}) = \lambda_1(z)\Delta(\gamma_{H_1}, \gamma_{G_1}), \quad z \in Z_1(F),$$

where λ_1 is the character on the center of $G_1(F)$ attached to **a** [see the beginning of the proof of (4.4.A)]. To conclude that

$$\Delta(z\gamma_{H_1}, \gamma_G) = \lambda(z)\Delta(\gamma_{H_1}, \gamma_G)$$

we have only to show that λ_1 coincides with λ on $Z_1(F)$. But **a** is represented by the cocycle a defined by

$$\xi_1 \cdot \xi_{T_{H_1}}(w) = a(w)\xi_{T_1}(w), \quad w \in W_F,$$

where $T_{H_1} = \mathrm{Cent}(\gamma_{H_1}, H_1)$ and $T_{H_1} \to T_1$ is an admissible embedding. To compute λ_1 on $Z_1(F)$ we project a onto $\widehat{Z}_1$, obtaining $a_1 : W_F \to \widehat{Z}_1$. Since, by construction, $\xi_{T_{H_1}}(w) \in \widehat{H} \times w$ and $\xi_{T_1}(w) \in \widehat{G} \times w, w \in W_F$, we have that $W_F \xrightarrow{\xi_1} {}^LG_1 \longrightarrow {}^LZ_1$ coincides with $w \to a_1(w) \times w$ and then $\lambda_1 = \lambda$ on $Z_1(F)$ by definition.

The group H_1 is determined up to isomorphism by G_1 and $(H, \mathcal{H}, s, \xi)$, but ξ_1 may be replaced by $b \otimes \xi_1$ where b is a 1-cocycle of W_F in the center of $\widehat{H}_1$. This cocycle determines a character λ_0 on $H_1(F)$ and λ, λ_1 are replaced by $\lambda_0\lambda, \lambda_0\lambda_1$.

Finally, we observe that it is only the equivalence class of $(H, \mathcal{H}, s, \xi)$ that matters for the definition of Δ. The choice of twist $\psi : G \to G^*$ does affect Δ, but ψ may be replaced by $\mathrm{Int}\, x \circ \psi, x \in G^*$, without effect.

5. Regular Unipotent Analysis

(5.1) *Regular Unipotent Elements (Review)*

Recall that the regular unipotent elements of $G(\overline{F})$ are characterized by the property that each lies in exactly one Borel subgroup of G. They form a single conjugacy class [St].

Suppose B is a Borel subgroup of G containing the maximal torus T. Denote by N the unipotent radical of B. For a simple root α of T in B let N_α be the 1-parameter subgroup of N attached to α and N^α be the subgroup of N generated by the 1-parameter subgroups for the remaining roots of T in B, so that N is the direct product $N_\alpha \cdot N^\alpha$. Set $N' = \bigcap_\alpha N^\alpha$. Given root vectors $\{X_\alpha\}$ we define $x_\alpha(u)$ for u in N by

$$u \equiv \exp x_\alpha(u) X_\alpha (\operatorname{mod} N^\alpha).$$

Then u is regular if and only if $x_\alpha(u)$ is nonzero for all simple α. See [St, pp. 110-112] for this and the next paragraph.

Fix u_0 regular in N. Then every regular element in N may be written in the form

$$u = t^{-1} u' u_0 t,$$

where $u' \in N'$ is uniquely determined and $t \in T$ is determined modulo the center Z of G. Conversely, every element of this form is regular in N. Note that $x_\alpha(t^{-1}u'u_0t) = \alpha(t)^{-1} x_\alpha(u_0)$.

Let u be any regular unipotent element of G, B_u be the Borel subgroup containing u, and T_u be a maximal torus in B_u. For each simple root α of T_u in B_u define a root vector X^u_α by requiring $\exp X^u_\alpha$ to be the projection of u onto N_α. Write **spl**(u) for the splitting $(B_u, T_u, \{X^u_\alpha\})$ of G. Every splitting of G is obtained in this manner.

The following are equivalent:

(i) G is quasi-split over F.

(ii) G has an F-splitting.

(iii) There are regular unipotent elements in $G(F)$.

For (ii) $\Rightarrow$ (ii) we observe that if $u \in G(F)$ then **spl**(u) is an F-splitting as long as T_u is chosen over F. Note also that then **spl**(u) is determined up to $N(F)$-conjugacy by u. For (i) $\Rightarrow$ (iii), the existence of a Borel subgroup over F implies that the conjugacy class of regular unipotent elements is defined over F. This class then contains an F-rational point [K1].

From now on G will be quasi-split over F. The regular unipotent elements in $G(F)$ form a single stable conjugacy class by definition [K1]. Suppose T and $B = TN$ are defined over F. Then each $G(F)$-conjugacy class of regular unipotent elements in $G(F)$ meets $N(F)$. If $u_0 \in N(F)$ is regular then the regular element $u = t^{-1}u'u_0t$ in N is F-rational if and only if $u' \in N'(F)$ and $t\sigma(t^{-1}) \in Z(\overline{F}), \sigma \in \Gamma$.

Lemma 5.1.A. *The correspondence $u \to \mathbf{spl}(u)$ induces a bijection between the $G(F)$-conjugacy classes of regular unipotent elements in $G(F)$ and the $G(F)$-conjugacy classes of F-splittings of G.*

Proof. As above, if $u \in B(F)$ then $\mathbf{spl}(u)$ is determined up to $N(F)$-conjugacy, where N is the unipotent radical of B. On the other hand, if $\mathbf{spl} = (B, T, \{X_\alpha\})$ is an F-splitting then we can find $u \in N$ such that $\mathbf{spl}(u) = \mathbf{spl}$. For $\sigma \in \Gamma$ we must have $\mathbf{spl}(\sigma u) = \mathbf{spl}$ as well, which implies that $u^{-1}\sigma(u)$ lies in N'. Because $H^1(\Gamma, N'(\overline{F}))$ is trivial we can find $u' \in N'$ such that $uu' \in N(F)$. Then $\mathbf{spl} = \mathbf{spl}(uu')$. It is clear now that $u \to \mathbf{spl}\, u$ induces a surjective map from $G(F)$-conjugacy classes to $G(F)$-conjugacy classes. For injectively it is enough to show that if $B = TN$ is over F and $\mathbf{spl}(u_1) = \mathbf{spl}(u_2)$, where u_1 and u_2 lie in $N(F)$, then u_1 and u_2 are $N(F)$-conjugate. But $\mathbf{spl}\, u_1 = \mathbf{spl}\, u_2$ implies that $u_1 = t^{-1}u_1'u_0t$ and $u_2 = t^{-1}u_2'u_0t$ with $t \in T$ and $u_1', u_2' \in N'$, for some fixed regular $u_0 \in N(F)$. Because $u_1, u_2 \in N(F)$ we have $t\sigma(t^{-1}) \in Z, \sigma \in \Gamma$, and $u_1', u_2' \in N'(F)$. Thus it is enough to show that $u_1'u_0$ and $u_2'u_0$ are conjugate under $N(F)$. This is so because $N'(\overline{F})u_0$ is the $N(\overline{F})$-conjugacy class of u_0 [St, p. 112] and $H^1(\Gamma, N(\overline{F})) = 1$. The proof is then complete.

We now define a transfer factor $\Delta(u)$ for u regular unipotent in $G(F)$. Fix an F-splitting $\mathbf{spl} = (\mathbf{B}, \mathbf{T}, \{X_\alpha\})$ of G. There exists $h \in G_{\mathrm{sc}}$ such that

$$\mathbf{spl}(u)^h = \mathbf{spl},$$

where h acts in the obvious manner. Then $h\sigma(h)^{-1}$ lies in the center Z_{sc} of $G_{\mathrm{sc}}, \sigma \in \Gamma$. The class $\mathrm{inv}(u)$ of $\sigma \to h\sigma(h)^{-1}$ in $H^1(\Gamma, Z_{\mathrm{sc}})$ is well-defined.

To pair $\mathrm{inv}(u)$ with the endoscopic datum s we choose any maximal torus T over F in G which contains regular elements with images in $H(F)$. Inv(u) has an image $\mathrm{inv}_T(u)$ under $H^1(Z_{\mathrm{sc}}) \to H^1(T_{\mathrm{sc}})$. As earlier, (3.1), s determines an element $\mathbf{s}_T$ of $\pi_0(\widehat{T}_{\mathrm{ad}}^\Gamma)$. We set

$$\langle \mathrm{inv}(u), s \rangle = \langle \mathrm{inv}_T(u), \mathbf{s}_T \rangle.$$

The argument used in the proof of Lemma 3.2.A shows that $\langle \mathrm{inv}(u), s \rangle$ is independent of the choice for T. In addition if we define $\Delta(\overline{\gamma}_H, \overline{\gamma}_G)$ and $\Delta_0(\overline{\gamma}_H, \overline{\gamma}_G)$ as in (3.7), but use – for reasons that will appear later – the opposite splitting $\mathbf{spl}_\infty = (\mathbf{B}_\infty, \mathbf{T}, \{X_{-\alpha}\})$, where $\mathbf{B}_\infty \cap \mathbf{B} = \mathbf{T}$ and the root vectors $X_{-\alpha}$ are fixed as in (2.1), then

$$\Delta(u) = \frac{\Delta(\overline{\gamma}_H, \overline{\gamma}_G)}{\Delta_0(\overline{\gamma}_H, \overline{\gamma}_G)} \langle \mathrm{inv}(u), s \rangle$$

is independent of the choice of $\mathbf{spl}$. Finally, $\Delta(u)$ depends only on the $G(F)$-conjugacy class of u in $G(F)$.

(5.2) *Stars and the Variety X*

This will be a review of some material from [L3]. We continue with G quasi-split over F and $(\mathbf{B}, \mathbf{T}, \{X_\alpha\})$ an F-splitting of G.

Let T be a maximal torus over F in G. We fix some Borel subgroup B_T containing T and in the usual manner transport the roots, Weyl group, Weyl chambers and Galois action for T to $\mathbf{T}$ without change in notation. We denote by Ω the Weyl group, by $\mathfrak{W}$ the set of Weyl chambers and by W_+ the chamber attached to B_T or $\mathbf{B}$. If $\omega \in \Omega$ we write $W(\omega)$ for the chamber $\omega^{-1}W_+$ and B_T^ω for the Borel subgroup $w^{-1}B_T w$, where $w \in G$ represents ω.

Let S be the variety of stars attached to T [L3]. The choice of B_T affects S, but a different choice yields an F-isomorphic variety. Recall that the elements of S are functions from $\mathfrak{W}$ to $\mathcal{B}$, the variety of Borel subgroups of G. A typical element will be denoted $(B(W))$. The F-structure is defined by

$$\sigma(B(W)) = (\sigma(B(\sigma_T^{-1}(W)))), \quad \sigma \in \Gamma,$$

and G acts on the right:

$$(B(W))^g = (g^{-1}B(W)g).$$

If W, W' are adjacent chambers, so that $W = W(\omega)$ and $W' = W(\omega(\alpha)\omega)$ for some $\omega \in \Omega$ and simple root α of $\mathbf{T}$ in $\mathbf{B}$ then by definition $(B(W), B(W'))$ lies in the closure of the orbit of $(B_T, B_T^{\omega(\alpha)})$ in $\mathcal{B}^2$.

The *standard* star s_0 is given by $B(W(\omega)) = B_T^\omega, \omega \in \Omega$. A star is *regular* if it lies in the G-orbit of s_0. The F-rational regular stars form the orbit of s_0 under $\mathfrak{A}(T) = \{g \in G(\overline{F}) : g\sigma(g^{-1}) \in T\}$.

Let $\mathbf{B}_\infty$ be opposite to $\mathbf{B}$ relative to $\mathbf{T}$. Then $S(\mathbf{B}_\infty)$ consists of all $(B(W))$ for which each $B(W)$ is opposite to $\mathbf{B}_\infty$, and $S(\mathbf{B}_\infty, \mathbf{B})$ consists of those stars for which we also have $B(W_+) = \mathbf{B}$. If $\mathbf{B}_\infty = \mathbf{T}\mathbf{N}_\infty$ then the morphism

$$(n, (B(W))) \to (B(W))^n$$

from $\mathbf{N}_\infty \times S(\mathbf{B}_\infty, \mathbf{B})$ to $S(\mathbf{B}_\infty)$ allows us to identify these two varieties.

If W is a chamber and β a W-simple root the coordinate function $z(W, \beta)$, or $z(\omega, \alpha)$, is defined on $S(\mathbf{B}_\infty, \mathbf{B})$. Here $\omega \in \Omega$ and $\mathbf{B}$-simple α are given by $W = W(\omega)$ and $\omega\beta = \alpha$. The chamber $W' = W(\omega(\alpha)\omega)$ is adjacent to W and there is a unique $h \in \mathbf{N}_\infty$ such that

$$hB(W)h^{-1} = \mathbf{B},$$

and then

$$hB(W')h^{-1} = \exp(-zX_{-\alpha})\mathbf{B}\exp zX_{-\alpha},$$

where $z = z(W, \beta) = z(\omega, \alpha)$ lies in $\overline{F}$. We have, for F-rational $(B(W))$,

$$\sigma(z(\omega, \alpha)) = z(\boldsymbol{\sigma}\omega\sigma_T^{-1}, \boldsymbol{\sigma}\alpha), \qquad \sigma \in \Gamma. \tag{5.2.1}$$

If $g \in G$ and $s = (B(W))$ is a star we write $g \in s$ if $g \in \bigcap_W B(W)$. Let X^0 consist of all pairs (g,s), where g is regular semisimple, s is regular and $g \in s$, and let X be the closure of X^0 in $G \times S$. Both X^0 and X are defined over F, and X is contained in $\{(g,s) : g \in s\}$. Thus $\xi : (g,(B(W))) \to (g, B(W_+))$ is a well-defined morphism from X to the Springer-Grothendieck variety $M = \{(g,B) : g \in B\}$. Let $\pi_M : M \to G$ be the projection onto the first factor and $\phi_M : M \to T$ be defined by $\phi_M(g,B) = \gamma$ if $h^{-1}gh \equiv \gamma \mod N_T$ where $B^h = B_T$ and N_T is the unipotent radical of B_T. Both π_M and ϕ_M are smooth and π_M is proper. Set $\pi = \pi_M \circ \xi$ and $\phi = \phi_M \circ \xi$. Then both π and ϕ are defined over F, and ϕ is smooth and proper.

Set $M^0 = \pi_M^{-1}(G_{\text{regss}})$. Then $\xi : X^0 \to M^0$ is an isomorphism. If $\gamma \in T$ is regular then $\phi^{-1}(\gamma)$ is the G-orbit of (γ, s_0) and so may be identified with the conjugacy class of γ in G. If also γ is F-rational then $\phi^{-1}(\gamma)(F)$ is identified with the stable conjugacy class of γ in $G(F)$.

(5.3) *Regular Unipotent Elements and* X

Suppose $u \in G$ is regular unipotent and contained in the Borel subgroup B_u. Then we define the star s_u by $B(W) = B_u, W \in \mathfrak{W}$, and set $x_u = (u, s_u)$.

Lemma 5.3.A.

(i) *x_u lies in X.*

(ii) *If $u \in G(F)$ then $x_u \in X(F)$.*

(iii) *$\xi : X \to M$ is invertible at x_u.*

Proof. A point in $S(\mathbf{B}_\infty)$ may be written s^h, where $s \in S(\mathbf{B}_\infty, \mathbf{B})$ and $h \in \mathbf{N}_\infty$. Suppose $g \in s^h$. Then we write $g = h^{-1}tnh$, where $t \in \mathbf{T}, n \in \mathbf{N}$. We calculate the z-coordinates of s in terms of t, n as follows.

Let $s = (B(W))$. If α is $\mathbf{B}$-simple then $z = z(W_+, \alpha)$ is the solution to

$$B(\omega(\alpha)W_+) = \exp(-zX_{-\alpha})\mathbf{B}\exp zX_{-\alpha}.$$

On the other hand, write tn as

$$tn = t\exp(x_\alpha(n)X_\alpha)n'$$

where $n' \in \mathbf{N}^\alpha$. The condition that

$$t\exp(x_\alpha(n)X_\alpha)n' \in B(\omega(\alpha)W_+)$$

is the requirement in $SL(2)$ that

$$\begin{bmatrix} 1 & 0 \\ z & 1 \end{bmatrix}\begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix}\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -z & 1 \end{bmatrix} = \begin{bmatrix} a_1 & x_1 \\ y_1 & b_1 \end{bmatrix}$$

be upper triangular, where $x = x_\alpha(n), z = z(W_+, \alpha)$ and $\alpha(t) = a^2$.

If s is regular then $z \neq 0$ and

$$xz = x_\alpha(n)z(W_+, \alpha) = 1 - a^{-2} = 1 - \alpha(t)^{-1}.$$

This equation continues to hold on X. Observe also that a_1 is then a^{-1} and x_1 is a^2x. More generally, suppose $W = W(\omega)$ and $\omega\beta = \alpha$. Then:

Lemma 5.3.B. *If n is regular then*

$$z(W, \beta) = \frac{1 - \beta(t)^{-1}}{x(W, \beta)},$$

where $x(W, \beta)$ is a rational function of tn which is defined and equal to $x_\alpha(n)$ at $t = 1$.

Proof. Consider now $z = z(\omega(\alpha_1)W_+, \omega(\alpha_1)\alpha_0)$, where α_0, α_1 are $\mathbf{B}$-simple. Let z_1 be the coordinate $z(W_+, \alpha_1)$ of s, so that

$$\exp z_1 X_{-\alpha_1} B(\omega(\alpha_1)W_+) \exp(-z_1 X_{-\alpha_1}) = \mathbf{B}$$

and

$$\exp z_1 X_{-\alpha_1} B(\omega(\alpha_1)\omega(\alpha_0)W_+) \exp(-z_1 X_{-\alpha_1}) = \exp(-zX_{-\alpha_0})\mathbf{B} \exp zX_{-\alpha_0}.$$

Thus z is the coordinate $z(W_+, \alpha_0)$ for the star

$$B_1(W) = \exp z_1 X_{-\alpha_1} B(\omega(\alpha_1)W) \exp(-z_1 X_{-\alpha_1}), \quad W \in \mathfrak{W},$$

which is regular if s is. Replace tn by

$$t_1 n_1 = \exp z_1 X_{-\alpha_1} tn \exp(-z_1 X_{-\alpha_1}).$$

Then if $x_{\alpha_0}(n_1) \neq 0$ we get

$$z(\omega(\alpha_1)W_+, \omega(\alpha_1)\alpha_0)\frac{1 - \alpha_0(t_1)^{-1}}{x_{\alpha_0}(n_1)}.$$

The earlier $SL(2)$ calculation shows that $t_1 = \omega(\alpha_1)(t)$ and also that if $\alpha_0 = \alpha_1$ then $x_{\alpha_0}(n_1) = \alpha_0(t)x_{\alpha_0}(n)$. Otherwise $x_{\alpha_0}(n_1)$ is a more complicated function of t and a but it is equal to $x_{\alpha_0}(n)$ at $t = 1$, for then $z_1 = 0$. Thus the lemma is verified in the case of $z(\omega(\alpha_1)W_+, \omega(\alpha_1)\alpha_0)$. We repeat this procedure to obtain the lemma in general.

From this lemma we deduce immediately Lemma 5.3.A. For $(h^{-1}tnh, s) \in X$ the star s is a rational function of t, n and h that is defined in a neighborhood of $t = 1, n = u$, where it takes the value s_u.

Corollary 5.3.C. *$\phi : X \to T$ is smooth at x_u.*

(5.4) *Orbital Integrals as Fiber Integrals*

From now on we assume F local and view $\sum_{\gamma_G} \Delta(\gamma_H, \gamma_G)\Phi(\gamma_G, f), f \in C_c^\infty(G(F))$, as a fiber integral on $X(F)$. Fix for once and for all an admissible embedding $T_H \to T$ of T_H in G, which is quasi-split over F, along with the Borel subgroup B_T containing T. If γ is the image of γ_H under $T_H \to T$ then

$$\gamma \to \sum \Delta(\gamma_H, \gamma_G)\phi(\gamma_G, f)$$

is a function on the regular elements of $T(F)$. Its value at γ is an integral over the fiber $\phi^{-1}(\gamma)(F)$ in $X(F)$ as follows.

The forms ω_G, ω_T on G, T and measures $|\omega_G|, |\omega_T|$ have been specified in (1.4). To ω_G there is attached a G-invariant form ω_M of highest degree on M nowhere vanishing on M^0 and hence, after transport by ξ, a G-invariant form ω_X of highest degree on X nowhere vanishing on X^0 [L3, Lemma 2.8]. We embed the variety $\mathcal{U}_{\text{reg}}$ of regular unipotent elements in G as an open subvariety of $\phi^{-1}(1)$ under $u \to x_u$. More generally if $z \in Z$ we embed $z\mathcal{U}_{\text{reg}}$ as an open subvariety of $\phi^{-1}(z)$. Then ϕ is smooth at the points in $z\mathcal{U}_{\text{reg}}$ and we see easily that ω_X is nonvanishing around $z\mathcal{U}_{\text{reg}}$. If $\gamma \in T(F)$ is regular then the quotient of ω_X by $\phi^*(\omega_T)$ defines a G-invariant form ω_γ of highest degree along $\phi^{-1}(\gamma)$. For $z \in Z$ we similarly obtain a G-invariant form ω_z of highest degree along $z\mathcal{U}_{\text{reg}}$. For γ and z F-rational the measures $|\omega_\gamma|$ on $\phi^{-1}(\gamma)(F)$ and $|\omega_z|$ on $Z_{\text{reg}}(F)$ are specified in the manner of (1.4). Recalling the definition of ω_T in (1.4) we note that $|\omega_z|$ is independent of T. On the other hand, by [L3, Lemma 2.12],

$$|\omega_\gamma| = \prod_\alpha |1 - \alpha(\gamma)^{-1}| \cdot |\omega_G|/|\omega_T|,$$

where the product is over roots α of T in B_T. We replace $|\omega_\gamma|$ by

$$|\omega_\gamma^*| = \prod_\alpha |\alpha(\gamma)|^{1/2} |\omega_\gamma|$$

to obtain

$$|\omega_\gamma^*| = D_G(\gamma)|\omega_G|/|\omega_T|.$$

Then for γ_H strongly G-regular we may write

$$D_H(\gamma_H) \sum_{\gamma_G} \Delta(\gamma_H, \gamma_G)\Phi(\gamma_G, f)$$

as

$$\int_{\phi^{-1}(\gamma)(F)} \Delta(x) f(\pi(x)) |\omega_\gamma^*|,$$

where

$$\Delta(x) = \Delta(\gamma_H, \pi(x)) D_H(\gamma_H) D_G(\gamma)^{-1}.$$

With suitable conventions there is a similar formula in the case γ is not strongly regular, but we shall not need this fact.

Since G is quasi-split over F we have defined $\Delta(\gamma_H, \gamma_G)$ as

$$\frac{\Delta(\overline{\gamma}_H, \overline{\gamma}_G)}{\Delta_0(\overline{\gamma}_H, \overline{\gamma}_G)} \Delta_0(\gamma_H, \gamma_G),$$

where $\overline{\gamma}_H, \overline{\gamma}_G$ are fixed and

$$\Delta_0 = \Delta_I \Delta_{II} \Delta_1 \Delta_2 \Delta_{IV}.$$

But $\Delta_{IV}(\gamma_H, \gamma_G) = D_G(\gamma) D_H(\gamma_H)^{-1}$ and $\Delta_I(\gamma_H, \gamma_G), \Delta_{II}(\gamma_H, \gamma_G), \Delta_2(\gamma_H, \gamma_G)$ depend only on γ. Thus $\Delta_1(x) = \Delta_1(\gamma_H, \pi(x))$ alone varies along the fiber $\phi^{-1}(\gamma)(F)$.

Write x as (γ_G, s) and assume $s \in S(\mathbf{B}_\infty)$. To specify $\Delta_1(x)$, we choose $g \in G$ such that

$$(\gamma_G, s) = (\gamma, s_0)^g,$$

where s_0 is the standard star. Then $\mathrm{inv}(\gamma_H, \gamma_G)$ is the class of the cocycle $\sigma \to g\sigma(g)^{-1}$ in $H^1(T)$ and

$$\Delta_1(x) = \langle \mathrm{inv}(\gamma_H, \gamma_G)^{-1}, \mathbf{s}_T \rangle$$

[see (3.4)]. More precisely, we should pass to G_{sc} to define $\mathrm{inv}(\gamma_H, \gamma_G)$. We do so without change in notation. Proposition 5.2 of [L3] describes the inverse cocycle $\sigma(g)g^{-1}$ in terms of coordinates. We recall this next.

The cocycle $\lambda(T)$ from (2.3) will be computed relative to a-data $\{a_\alpha\}$ and the splitting *opposite* to $(\mathbf{B}, \mathbf{T}, \{X_\alpha\})$. Also p will denote the gauge on the roots of T in G attached to B_T, and $\prod\limits_{1,\sigma}^{p}$ will indicate a product over roots α such that $p(\alpha) = 1$ and $p(\sigma_T^{-1}\alpha) = -1$. Suppose $\omega(\alpha_1) \ldots \omega(\alpha_r)$ is a reduced expression for $\omega_T(\sigma)$, where $\sigma_T = \omega_T(\sigma) \rtimes \boldsymbol{\sigma}$. Set $\omega_0 = 1$ and $\omega_k = \omega(\alpha_1) \ldots \omega(\alpha_k), 1 \leqq k \leqq r$. If $p(\alpha) = 1$ and $p(\sigma_T^{-1}\alpha) = -1$ then $\alpha = \omega_{k-1}(\alpha_k)$, some $1 \leqq k \leqq r$, and we may set

$$z(\sigma, \alpha) = z(-\omega_{k-1} W_+, -\alpha).$$

Note that $-\omega_{k-1} W_+ = \omega_{k-1} \omega_- W_+$ and

$$-\alpha = \omega_{k-1} \omega_- \overline{\alpha}_k,$$

where $\omega_- \in \boldsymbol{\Omega}$ maps W_+ to $-W_+$ and $\overline{\alpha}_k = -\omega_- \alpha_k$.

Lemma 5.4.A. $\mathrm{inv}(\gamma_H, \gamma_G)^{-1}$ *is represented by the cocycle*

$$\sigma \to \lambda(T)^{-1} \prod_{1,\sigma}^{p} \left(\frac{-a_\alpha}{z(\sigma, \alpha)} \right)^{\alpha^\vee}.$$

Proof. This formula has just to be reconciled with that of [L3, Proposition 5.2]. We have used h, where $(B_T, T)^h = (\mathbf{B}, \mathbf{T})$, to identify roots of T with roots of $\mathbf{T}$. Suppose that $(B_T, T)^{h_1} = (\mathbf{B}_\infty, \mathbf{T})$ and $\omega_\infty(\sigma) = \omega_- \omega_T(\sigma)\omega_-$. Then

$$\omega_\infty(\sigma) = \omega(\overline{\alpha}_1)\dots\omega(\overline{\alpha}_r)$$

is a reduced expression. We define $n(\omega_\infty(\sigma))$ and $n(\omega_\infty(\sigma)^{-1})$ as in (2.1), but relative to the splitting *opposite* to $(\mathbf{B}, \mathbf{T}, \{X_\alpha\})$. Thus

$$n(\omega_\infty(\sigma)^{-1}) = n(-\overline{\alpha}_r)\dots n(-\overline{\alpha}_1).$$

By Lemma 2.1.A

$$n(\omega_\infty(\sigma)^{-1}) = n(\omega_\infty(\sigma))^{-1} \prod\nolimits_\infty (-1)^{\alpha^\vee},$$

where $\prod_\infty$ indicates a product over roots α such that both α and $-\omega_\infty(\sigma)^{-1}\alpha$ are positive for $\mathbf{B}_\infty$. Note that $h_1(\prod_\infty (-1)^{\alpha^\vee})h_1^{-1} = \prod\limits_{1,\sigma}^{p} (-1)^{\alpha^\vee}$. Applying (5.2.1) we rewrite the formula of [L3] as

$$\sigma(h_1)n(\omega_\infty(\sigma))^{-1} \prod\nolimits_\infty (-1)^{\alpha^\vee} h_1^{-1}$$

times

$$h_1 \left(\prod_{k=1}^{r} z(\sigma, \omega_{k-1}(\alpha_k))^{\omega(\overline{\alpha}_1)\dots\omega(\overline{\alpha}_{k-1})\overline{\alpha}_k^\vee} \right) h_1^{-1}.$$

(To make the comparison with [L3] easier we note that $h_1 \leftrightarrow hw_-^{-1}$, that $n(w_\infty(\sigma)^{-1}) \leftrightarrow w_{a_j}\dots w_{\alpha_1}$, and that all u_{α_i} are 1. We recall from (5.2.1) that

$$\sigma(z(w, \alpha)) = z(\sigma\omega\sigma_T^{-1}, \boldsymbol{\sigma}\alpha).)$$

Since $\lambda(T) : \sigma \to h_1(\prod_\infty a_\alpha^{\alpha^\vee})n(\omega_\infty(\sigma))\sigma(h_1^{-1})$, we obtain

$$\lambda(T)^{-1} \prod_{1,\sigma}^{p} a_\alpha^{\alpha^\vee} \cdot \prod_{1,\sigma}^{p} (-1)^{\alpha^\vee} \prod_{1,\sigma}^{p} z(\sigma, \alpha)^{-\alpha^\vee},$$

as desired, because $\operatorname{Int} h_1$ takes the root $\overline{\alpha}$ of $\mathbf{T}$ to the root $-\alpha$ of T.

Lemma 5.4.B. *If $\gamma_G = n_1^{-1}tnn_1$ where $n \in \mathbf{N}$ is regular and $n_1 \in \mathbf{N}_\infty$ then $z(\sigma, \alpha) = \frac{1-\alpha(\gamma)}{x(\sigma,\alpha)}$, where $x(\sigma, \alpha) \to x_{\overline{\alpha}_k}(n)$ as $\gamma \to 1$.*

Proof. $\phi(\gamma_G, s) = \gamma$ implies that $h^{-1}\gamma h = t$ and we have only to apply Lemma 5.3.B to $z(-\omega_{k-1}W_+, -\alpha)$.

The constructions and results of (5.3) and (5.4) are described for $SL(2)$ in [L-S].

(5.5) *A Limit Formula*

We assume that H_1 is a central extension of H as constructed in (4.4). The character λ^{G_1} of Lemma 4.4.A defines characters λ on $Z_1(F)$, the kernel of $H_1(F) \to H(F)$, and λ^G on $Z_1^G(F)$, the preimage of the center $Z(F)$ of

$G(F)$ under $H_1(F) \to H(F)$. Recall that $Z(F)$ is canonically embedded in $H(F)$. The projection of $Z_1^G(F)$ onto $Z(F)$ will be written as $z_1 \to z$, and γ_1 will be an element of $H_1(F)$ with strongly G-regular image γ_H in $H(F)$.

Theorem 5.5.A. $\lim\limits_{\gamma_1 \to z_1} D_{H_1}(\gamma_1) \sum\limits_{\gamma_G} \Delta(\gamma_1, \gamma_G)\Phi(\gamma_G, f)$

is equal to

$$\lambda^G(z_1) \sum_u \Delta(u)\Phi(zu, f), \quad f \in C_c^\infty(G(F)),$$

where $\sum\limits_u$ *indicates summation over representatives* u *for the* $G(F)$*-conjugacy classes of regular unipotent elements in* $G(F)$*, and*

$$\Phi(zu, f) = \int f|\omega_z|,$$

the integral being taken over the conjugacy class of zu*, an open subset of* $z\mathcal{U}_{\mathrm{reg}}(F)$*.*

Corollary 5.5.B. *If* $f \in C_c^\infty(G(F))$ *and* $f^1 \in C_c^\infty(H_1(F), \lambda)$ *have* Δ*-matching orbital integrals then*

$$\sum_{u_1} \Phi(z_1 u_1, f^1) = \lambda^G(z_1) \sum_u \Delta(u)\Phi(zu, f).$$

Proof of the Theorem. Because $\Delta(z_1\gamma_1, z\gamma_G) = \lambda^G(z_1)\Delta(\gamma_1, \gamma_G)$ we reduce immediately to the case $z_1 = 1$. Replacing G by G_1 we may assume that $H_1 = H$. By known properties of the asymptotic behavior of orbital integrals it suffices to consider a function f supported in a small neighborhood of a regular unipotent element in $G(F)$.

Suppose that γ is strongly regular, and let $x = (\gamma_G, s)$ lie in $\phi^{-1}(\gamma)(F)$, with x in the coordinate patch $S(\mathbf{B}_\infty)$. We write x as $(tn, s_1)^{n_1}$, where $n_1 \in \mathbf{B}_\infty, s_1 \in S(\mathbf{B}_\infty, \mathbf{B})$ and $t = h^{-1}\gamma h$, and assume $n \in \mathbf{N}$ is regular. Then all that has to be shown is that as $\gamma \to 1$ we have $\Delta(x) \to \Delta(n)$. From (5.4) we have

$$\Delta(x) = \frac{\Delta(\overline{\gamma}_H, \overline{\gamma}_G)}{\Delta_0(\overline{\gamma}_H, \overline{\gamma}_G)} \cdot \Delta_1(\gamma_H, \gamma_G) \cdot \Delta_{II}(\gamma_H, \gamma_G) \cdot \Delta_1(\gamma_H, \gamma_G) \cdot \Delta_2(\gamma_H, \gamma_G).$$

Both $\Delta_0(\overline{\gamma}_H, \overline{\gamma}_G)$ and $\Delta_I(\gamma_H, \gamma_G)$ are to be computed relative to the same F-splitting, that *opposite* to $(\mathbf{B}, \mathbf{T}, \{X_\alpha\})$. Nothing else depends on a splitting.

The quotient $\Delta(\overline{\gamma}_H, \overline{\gamma}_G)/\Delta_0(\overline{\gamma}_H, \overline{\gamma}_G)$ is a constant which also appears in $\Delta(n)$. By definition

$$\Delta_I(\gamma_H, \gamma_G) = \langle \lambda(T), \mathbf{s}_T \rangle.$$

Also

$$\Delta_{II}(\gamma_H, \gamma_G) = \prod_\alpha \chi_\alpha \left(\frac{\alpha(\gamma) - 1}{a_\alpha} \right),$$

where the product is over representatives α for the Γ-orbits of roots of T outside H. If $\mathcal{O}$ is an asymmetric orbit then the contribution for $\pm\mathcal{O}$ is a character (which we could take to be trivial) evaluated at γ (Lemma 3.3.A).

Thus we need take into account only the symmetric orbits. It remains to consider $\Delta_1(\gamma_H, \gamma_G)$, for $\Delta_2(\gamma_H, \gamma_G)$ is a character evaluated at γ and so has limit 1. But

$$\Delta_1(\gamma_H, \gamma_G) = \langle \mathrm{inv}(\gamma_H, \gamma_G)^{-1}, \mathbf{s}_T \rangle$$

and $\mathrm{inv}(\gamma_H, \gamma_G)^{-1}$ is represented by the cocycle

$$\lambda(T)^{-1} \prod_{1,\sigma}^{p} \left(\frac{-a_\lambda}{z(\sigma, \alpha)} \right)^{\alpha^\vee}$$

of Lemma 5.4.A. On cancellation with Δ_I we may replace $\mathrm{inv}(\gamma_H, \gamma_G)^{-1}$ by the class of the cocycle

$$\prod_{1,\sigma}^{p} \left(\frac{-a_\alpha}{z(\sigma, \alpha)} \right)^{\alpha^\vee} = \prod_{1,\sigma}^{p} \left(\frac{a_\alpha x(\sigma, \alpha)}{\alpha(\gamma) - 1} \right)^{\alpha^\vee}$$

(see Lemma 5.4.B). But around 1, $\alpha(\gamma)^{1/2}$ is well defined and continuous. Thus we rewrite the cocycle as

$$\prod_{1,\sigma}^{p} \left(\frac{a_\alpha}{\alpha(\gamma)^{1/2} - \alpha(\gamma)^{-1/2}} \right)^{\alpha^\vee} \prod_{1,\sigma}^{p} \left(\frac{x(\sigma, \alpha)}{\alpha(\gamma)^{1/2}} \right)^{\alpha^\vee} .$$

Since $a_\alpha / (\alpha(\gamma)^{1/2} - \alpha(\gamma)^{-1/2})$ lies in $F_{\pm\alpha}^\times$ the first product is a cocycle (Lemma 2.2.B). The pairing of this cocycle with $\mathbf{s}_T$ yields

$$\prod_{\alpha} \chi_\alpha \left(\frac{a_\alpha}{\alpha(\gamma)^{1/2} - \alpha(\gamma)^{-1/2}} \right),$$

where the product is taken over representatives α for the symmetric Γ-orbits of roots outside H (Lemma 3.2.D). The product of this term with the contribution of the symmetric orbits to Δ_{II} is then:

$$\prod_{\alpha} \chi_\alpha \left(\frac{\alpha(\gamma) - 1}{\alpha(\gamma)^{1/2} - \alpha(\gamma)^{-1/2}} \right) = \prod_{\alpha} \chi_\alpha(\alpha(\gamma)^{1/2}),$$

and so approaches 1 as $\gamma \to 1$. We conclude that

$$\lim_{\gamma \to 1} \Delta(x) = \frac{\Delta(\overline{\gamma}_H, \overline{\gamma}_G)}{\Delta_0(\overline{\gamma}_H, \overline{\gamma}_G)} \lim_{\gamma \to 1} \langle \mathbf{C}(\gamma), \mathbf{s}_T \rangle,$$

where $\mathbf{C}(\gamma)$ is represented by the cocycle

$$\prod_{1,\sigma}^{p} (x(\sigma, \alpha)\alpha(\gamma)^{-1/2})^{\alpha^\vee} .$$

But $\lim_{\gamma \to 1} x(\sigma, \alpha)\alpha(\gamma)^{-1/2} = x_{\overline{\alpha}_k}(n)$ (Lemma 5.4.B). Thus it remains to show that $\prod_{1,\sigma}^{p} x_{\overline{\alpha}_k}(n)^{\alpha^\vee}$ represents the class $\mathrm{inv}_T(n)$ defined in (5.1).

We shall pass to $G_{\rm sc}$ as necessary, without change in notation. Choose $\mathbf{t} \in \mathbf{T}$ such that

$$\alpha(\mathbf{t}) = x_\alpha(n)$$

for all simple roots α. Then $\mathrm{inv}(n)$ is the class of $\sigma \to \mathbf{t}\sigma(\mathbf{t})^{-1}$ in $H^1(Z)$. Its image $\mathrm{inv}_T(n)$ in $H^1(T)$ is, after transport to $\mathbf{T}$, the class of this same cocycle but now for the σ_T-action on $\mathbf{T}$.

Set $\mathbf{t}_1 = \omega_-(\mathbf{t}^{-1})$, and note that

$$\alpha(\mathbf{t}_1) = x_{\overline{\alpha}}(n)$$

for all simple α. We have also that

$$\sigma(\mathbf{t}_1)\mathbf{t}_1^{-1} = \omega_-(\sigma\mathbf{t}^{-1})\omega_-(\mathbf{t}) = \omega_-(\mathbf{t}\sigma(\mathbf{t}^{-1})) = \mathbf{t}\sigma(\mathbf{t}^{-1})$$

since this last element is central. Thus it is enough to show that:

$$\prod_{1,\sigma}^{p} x_{\overline{\alpha}_k}(n)^{\alpha^\vee} \textit{ is cohomologous to } \sigma(\mathbf{t}_1)\mathbf{t}_1^{-1}.$$

But $\prod\limits_{1,\sigma}^{p} x_{\overline{\alpha}_k}(n)^{\alpha^\vee} = \prod\limits_{k=1}^{n} x_{\overline{\alpha}_k}(n)^{\omega_{k-1}(\alpha_k^\vee)}$. For this, recall that $\omega_T(\sigma) = \omega(\alpha_1)\ldots\omega(\alpha_r)$ is a reduced expression and $\omega_0 = 1, \omega_k = \omega(\alpha_1)\ldots\omega(\alpha_k), 1 \leqq k \leqq r$. Hence

$$\prod_{1,\sigma}^{p} x_{\overline{\alpha}_k}(n)^{\alpha^\vee} = \prod_{k=1}^{n} \alpha_k(\mathbf{t}_1)^{\omega_{k-1}(\alpha_k^\vee)} = \mathbf{t}_1\omega(\alpha_1)\ldots\omega(\alpha_r)(\mathbf{t}_1^{-1}) = \mathbf{t}_1\omega_r(\sigma)(\mathbf{t}_1^{-1}).$$

Since $\sigma\mathbf{t}_1 \equiv \mathbf{t}_1 (\mathrm{mod}\, Z)$, this equals

$$\sigma(\mathbf{t}_1)\omega_T(\sigma)(\sigma\mathbf{t}_1^{-1}) = \sigma(\mathbf{t}_1)\sigma_T(\mathbf{t}_1^{-1}) = \sigma(\mathbf{t}_1)\mathbf{t}_1^{-1}\cdot\mathbf{t}_1\sigma_T(\mathbf{t}_1^{-1}),$$

and we are done.

6. Global consequences

(6.1) *Outline*

Here the results take a simple form. To explain them we continue the example of $SL(2)$ from (1.1). Now F is a number field with adele ring $\mathbf{A}$. The global matching concerns

$$\sum_{\gamma_G} \Delta_{\mathbf{A}}(\gamma_H, \gamma_G)\Phi_{\mathbf{A}}(\gamma_G, f), \tag{6.1.1}$$

where $\gamma_H \neq \pm 1$ lies in $H(F)$ and is an *adelic image* of $\gamma_G \in G(\mathbf{A})$, $\Phi_{\mathbf{A}}(\gamma_G, f)$ is the integral of a function f on $G(\mathbf{A})$ along the $G(\mathbf{A})$-conjugacy class of γ_G, and the sum is over representatives γ_G for $G(\mathbf{A})$-conjugacy classes. The factor $\Delta_{\mathbf{A}}$ is prescribed as follows.

To say that γ_H is an adelic image of γ_G we mean, in this example, that γ_G is everywhere locally stably conjugate to the image γ of γ_H under some admissible embedding $H \to T$ of H in G defined over F. If $\gamma_{G,v}$ is the component of γ_G at the place v then the element $\mathrm{inv}(\gamma_H, \gamma_{G,v})$ of $H^1(\Gamma_v, T(E_v))$ is defined as in (III$_1$) of (1.1). For almost all v it is trivial and the image $\mathrm{inv}(\gamma_G)$ of $\sum_v \mathrm{inv}(\gamma_H, \gamma_{G,v})$ in the 2-element group $H^1(\Gamma, T(E)\backslash T(\mathbf{A}_E))$ is independent of the choice of γ. Moreover, $\mathrm{inv}(\gamma_G)$ is trivial if and only if the $G(\mathbf{A})$-conjugacy class of γ_G meets $G(F)$. Tate-Nakayama duality allows us to pair $\mathrm{inv}(\gamma_G)$ with the image s_T in $\widehat{T}^\Gamma$ of the endoscopic datum s, that is, with the nontrivial element of $\widehat{T}^\Gamma$. Then we define

$$\Delta_{\mathbf{A}}(\gamma_H, \gamma_G) = \langle \mathrm{inv}(\gamma_G), s_T \rangle.$$

Thus $\Delta_{\mathbf{A}}(\gamma_H, \gamma_G) = 1$ if the $G(\mathbf{A})$-conjugacy class of γ_G meets $G(F)$ and $\Delta_{\mathbf{A}}(\gamma_H, \gamma_G) = -1$ otherwise.

We use unnormalized Tamagawa measures to specify orbital integrals. The function f is to be of the form $\prod_v f_v$ where $f_v \in C_c^\infty(G(F_v))$ for all v, and for almost all v the function f_v is to be the characteristic function of $K_v = G(\mathcal{O}_v)$ divided by the measure of K_v. Then $\Phi_{\mathbf{A}}(\gamma_G, f) = \prod_v \Phi(\gamma_{G,v}, f_v)$. The local factor $\Delta(\gamma_H, \gamma_{G,v})$ was defined in (1.1). Inspection of the terms shows that

(6.1.2) $$\Delta(\gamma_H, \gamma_{G,v}) = 1 \quad \text{for almost all } v$$

and

(6.1.3) $$\prod_v \Delta(\gamma_H, \gamma_{G,v}) = \Delta_{\mathbf{A}}(\gamma_H, \gamma_G)$$

provided the fixed elements $\overline{\gamma}_H, \overline{\gamma}_G$ are F-rational and at each place v the otherwise arbitrary $\Delta(\overline{\gamma}_H, \overline{\gamma}_G)$ is chosen so that (6.1.2) and (6.1.3) are satisfied.

Then

$$\begin{aligned}\sum_{\gamma_G} \Delta_{\mathbf{A}}(\gamma_H, \gamma_G)\Phi_{\mathbf{A}}(\gamma_G, f) &= \sum_{\gamma_G} \prod_v \Delta(\gamma_H, \gamma_{G,v})\Phi(\gamma_{G,v}, f_v) \\ &= \prod_v \sum_{\gamma_{G,v}} \Delta(\gamma_H, \gamma_{G,v})\Phi(\gamma_{G,v}, f_v)\end{aligned}$$

by [L2, Lemma 8.3]. This equals $\prod_v f_v^H(\gamma_H)$ where f_v^H is the smooth extension of

$$\gamma_H \to \sum_{\gamma_{G,v}} \Delta(\gamma_H, \gamma_{G,v})\Phi(\gamma_{G,v}, f_v)$$

to $H(F_v)$ (see Lemma 1.1.A). For almost all v, f_v^H is the characteristic function of the maximal compact subgroup of $H(F_v)$ divided by its measure.

We set $f^H = \prod_v f_v^H$ to conclude that

$$f^H(\gamma_H) = \sum_{\gamma_G} \Delta_{\mathbf{A}}(\gamma_H, \gamma_G)\Phi_{\mathbf{A}}(\gamma_G, f).$$

This is the global matching of orbital integrals for our example (see [L-L, p. 756]).

In general we shall define an adelic factor $\Delta_{\mathbf{A}}$, and verify (6.1.2) and the product formula (6.1.3) for the local factors of Sect. 3. Then suppose that $f = \prod_v f_v, f^H = \prod_v f_v^H$ are as usual, and that strongly G-regular $\gamma_H \in H(F)$ is an adelic image of $\gamma_G \in G(\mathbf{A})$. Because of (6.1.2) and [L2, Lemma 8.3] we have for almost all v that

$$\Phi^{\mathrm{st}}(\gamma_H, f_v^H) = \sum_{\gamma_{G,v}} \Delta(\gamma_H, \gamma_{G,v}) \Phi(\gamma_{G,v}, f_v),$$

each side having non-zero contribution from only one conjugacy class. *Thus if f_v and f_v^H have Δ-matching orbital integrals for all v we conclude that*

$$\Phi^{\mathrm{st}}_{\mathbf{A}}(\gamma_H, f^H) = \sum_{\gamma_G} \Delta_{\mathbf{A}}(\gamma_H, \gamma_G) \Phi_{\mathbf{A}}(\gamma_G, f), \tag{6.1.4}$$

where the left side is, by definition, the sum of the integrals of f^H along the $H(\mathbf{A})$-conjugacy classes of elements everywhere locally stably conjugate to γ_H. With a little care this extends to elements γ_H which are G-regular but not strongly G-regular (recall (4.3) for the local analogue).

Finally we shall observe that the Global Hypothesis of [L2] is satisfied. Thus, assuming the Hasse Principle for G_{sc}, the factors of Sect. 3 will be correct for stabilization of the Arthur-Selberg Trace Formula (see [L2, Chap. VIII]).

(6.2) *Notation*

If v is a place of F then we fix an extension **v** of v to $\overline{F}$ and for $L \subseteq F$ denote by L_v the completion of L so determined. There will be no harm in working with some suitably large finite Galois extension $L \subset \overline{F}$ of F. Thus $\Gamma = \mathrm{Gal}(L/F)$ and $\Gamma_v = \mathrm{Gal}(L_v/F_v)$. Set $W = W_{L/F}$ and $W_v = W_{L_v/F_v}$. Then we fix $W_v \to W$ such that

$$\begin{array}{ccccccccc} 1 & \to & L_v^\times & \to & W_v & \to & \Gamma_v & \to & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \to & C_L & \to & W & \to & \Gamma & \to & 1 \end{array}$$

is commutative.

Global endoscopic data $(H, \mathcal{H}, s, \xi)$ yield data $(H, \mathcal{H}_v, \xi_v)$ for G as group over F_v such that each of

$$\begin{array}{ccccccccc} 1 & \to & \widehat{H} & \to & \mathcal{H}_v & \to & W_v & \to & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \to & \widehat{H} & \to & \mathcal{H} & \to & W & \to & 1 \end{array}$$

and

$$\begin{array}{ccc} \mathcal{H}_v & \xrightarrow{\xi_v} & {}^L G_v \\ \downarrow & & \downarrow \\ \mathcal{H} & \xrightarrow{\xi} & {}^L G \end{array}$$

is commutative, where ${}^L G_v$ is the semidirect product $\widehat{G} \rtimes W_v$ with W_v acting through $W_v \to W \xrightarrow{\varrho} \mathrm{Aut}\, \widehat{G}$.

Again $\mathcal{H}$ need not be an L-group and we introduce central extensions of H satisfying global analogues of the conditions of (4.4) (see [L1, K1]). In the following discussion we assume that H itself satisfies these conditions and identify $\mathcal{H}$ with ${}^LH$. Passage to the general case is then routine, following (4.4) for the local case.

Let $K = \prod_v K_v, K^* = \prod_v K_v^*$ be compact open subgroups of $G(\mathbf{A}_f), G^*(\mathbf{A}_f)$. We fix a finite set V_0 of places such that for $v \notin V_0$ the groups G, G^* are unramified at v and K_v, K_v^* are hyperspecial maximal compact subgroups of $G(F_v), G^*(F_v)$. Outside V_0, ψ is defined over F_v, up to composition with an inner automorphism which does not affect transfer factors. Choose $g_v \in G(\overline{F}_v)$ such that $\psi_v = \mathrm{Int}\, g_v \circ \psi$ is defined over F_v and takes K_v to K_v^*. Then ψ_v may be used to identify $G(F_v)$ with $G^*(F_v)$.

(6.3) *Adelic Images and Transfer Factors*

Suppose $\gamma_H \in H(F)$ is G-regular and lies in the maximal torus T_H of H.

At each place v we shall allow only admissible embeddings of T_H in G^* which are defined over F. These exist, by Steinberg's Theorem (see [K1]). Then we say that γ_H is an *adelic image* of $\gamma_G \in G(\mathbf{A})$ if, for every v, γ_H is an image of the component $\gamma_{G,v}$ of γ_G in $G(F_v)$. Thus if T_H maps to T and γ_H to $\gamma \in T(F)$ under some admissible embedding over F then we require that for each v there exists $x_v \in G^*(L_v)$ such that $\mathrm{Int}\, x_v \circ \psi$ maps the maximal torus in T containing $\gamma_{G,v}$ to T over F_v and carries $\gamma_{G,v}$ to γ.

Suppose $\gamma_H, \overline{\gamma}_H$ are strongly G-regular elements of $H(F)$ and are adelic images of $\gamma_G, \overline{\gamma}_G \in G(\mathbf{A})$. For an L splitting T the element

$$\mu_v = \mathrm{inv}\left(\frac{\gamma_H, \gamma_{G,v}}{\overline{\gamma}_H, \overline{\gamma}_{G,v}}\right)$$

of $H^1(\Gamma_v, U(L_v))$ was defined in (3.4).

Lemma 6.3.A. $\mu_v = 1$ *for almost all* v.

Proof. Take an L splitting T. For almost all $v \notin V_0$: (i) L is unramified at v; (ii) $\gamma, \gamma_{G,v}$ lie in K_v; and (iii) for each root α of T in $G^*, \alpha(\gamma)$ is a unit in the ring of integers of L_v. That γ_H is an image of $\gamma_{G,v}$ means, for $v \notin V_0$, that $\gamma_{G,v}$ and γ are stably conjugate. If (i), (ii), and (iii) are satisfied $\gamma_{G,v}$ and γ are conjugate under K_v ([L2, Lemma 8.3]). If K_v is the stabilizer of the hyperspecial point x of the Bruhat-Tits building of $G(F_v)$ we denote by K_{sc} the stabilizer of x in $G_{\mathrm{sc}}(F_v)$ and by $K_{\mathrm{sc},L}$ the stabilizer in $G_{\mathrm{sc}}(L_v)$. By [K3, (3.3.4)], $\gamma_{G,v}$ and γ are conjugate under $K_{\mathrm{sc},L}$. Since $H^1(\Gamma_v, T_{\mathrm{sc}} \cap K_{\mathrm{sc},L}) = 1$ for almost all v and $\gamma_{G,v}, \gamma$ are strongly regular we conclude that $\gamma_{G,v}, \gamma$ are conjugate under K_{sc} for almost all v. Then the class $\mathrm{inv}(\gamma_H, \gamma_{G,v})$ in $H^1(\Gamma_v, T_{\mathrm{sc}}(L_v))$ is trivial.

We argue similarly for $\overline{\gamma}_H$, and $\overline{\gamma}_G$ to obtain for almost all $v \notin V_0$ that

$$\mu_v = \mathrm{inv}\left(\frac{\gamma_H, \gamma_{G,v}}{\overline{\gamma}_H, \overline{\gamma}_{G,v}}\right) = \frac{\mathrm{inv}(\overline{\gamma}_H, \overline{\gamma}_{G,v})}{\mathrm{inv}(\gamma_H, \gamma_{G,v})} = 1,$$

and the lemma is proved.

Let μ be the image of $\Sigma_v \mu_v$ in $H^1(\Gamma, U(L)\backslash U(\mathbf{A}_L))$ under

$$\sum_v H^1(\Gamma_v, U(L_v)) \to H^1(\Gamma, U(L)\backslash U(\mathbf{A}_L))$$

given by $1 \to U(L) \to U(\mathbf{A}_L) \to U(L)\backslash U(\mathbf{A}_L) \to 1$ and the isomorphism $\Sigma_v H^1(\Gamma_v, U(L_v)) \to H^1(\Gamma, U(\mathbf{A}_L))$. The endoscopic datum s determines $\mathbf{s}_{U,v} \in \pi_0(\widehat{U}^{\Gamma_v})$ as in (3.4). By its definition s also determines $\mathbf{s}_T \in \pi_0(\widehat{T}_{\mathrm{ad}}^{\Gamma})$ and similarly $\mathbf{s}_{\overline{T}}$ [see (1.2), (3.1)]. As in (3.4) we may define $\mathbf{s}_U \in \pi_0(\widehat{U}^{\Gamma})$ which depends only on the choice of embeddings $T_H \to T, \overline{T}_H \to \overline{T}$. Global Tate-Nakayama duality allows us to pair μ with $\mathbf{s}_U$ and the local-global relationship for the pairing yields:

$$\langle \mu, \mathbf{s}_U \rangle = \prod_v \langle \mu_v, \mathbf{s}_{U,v} \rangle. \tag{6.3.1}$$

There is another way to define μ. Strongly G-regular γ_H is an adelic image of $\gamma_G \in G(\mathbf{A})$ if and only if there exists $h \in G^*_{\mathrm{sc}}(\mathbf{A}_L)$ such that

$$h\psi(\gamma_G)h^{-1} = \gamma$$

[see the proof of (6.3.A)]. We proceed as in the local case (3.4). Recall that $\psi\sigma(\psi)^{-1} = \mathrm{Int}\, u(\sigma), u(\sigma) \in G^*_{\mathrm{sc}}(L)$, and then $v(\sigma) = hu(\sigma)\sigma(h)^{-1}$ lies in T_{sc}, with $\partial v = \partial u$ taking values in $T_{\mathrm{sc}}(L)$. Thus $v(\sigma)$ defines an element μ_T of $H^1(\Gamma, T_{\mathrm{sc}}(L)\backslash T_{\mathrm{sc}}(\mathbf{A}_L))$. By global Tate-Nakayama duality we may pair μ_T with $\mathbf{s}_T$. Further, $\langle \mu_T, \mathbf{s}_T \rangle$ is the independent of the choice of admissible embedding $T_H \to T$ over F, and clearly

$$\langle \mu, \mathbf{s}_U \rangle = \langle \mu_{\overline{T}}, \mathbf{s}_{\overline{T}} \rangle / \langle \mu_T, \mathbf{s}_T \rangle. \tag{6.3.2}$$

It will be more convenient to write $\langle \mu_T, \mathbf{s}_T \rangle$ as $d(\gamma_H, \gamma_G)$.

Lemma 6.3.B.

(i) $d(\gamma'_H, \gamma_G) = d(\gamma_H, \gamma_G)$ *if* γ'_H *is stably conjugate to* γ_H *in* $H(F)$.

(ii) $d(\gamma_H, \gamma'_G) = d(\gamma_H, \gamma_G)$ *if* γ'_G *is* $G(\mathbf{A})$*-conjugate to* γ_G.

(iii) $d(\gamma_H, \gamma_G) = d(\overline{\gamma}_H, \overline{\gamma}_G)$ *if* $\gamma_H, \overline{\gamma}_H$ *are adelic images of* $\gamma_G, \overline{\gamma}_G \in G(F)$.

Proof. (i) is immediate. For (ii) we use (6.3.2) and then (6.3.1) to reduce to the proof of Lemma 4.1.C for the local case. For (iii), μ is trivial if $\gamma_G, \overline{\gamma}_G \in G(F)$ for then we find $h, \overline{h} \in G^*_{\mathrm{sc}}(L)$ such that $h\psi(\gamma_G)h^{-1} = \gamma, \overline{h}\psi(\overline{\gamma}_G)\overline{h}^{-1} = \overline{\gamma}$. (6.3.2) now yields the result.

Fix strongly G-regular $\overline{\gamma}_H \in H(F)$ and $\overline{\gamma}_G \in G(F)$ such that $\overline{\gamma}_H$ is an adelic image of $\overline{\gamma}_G$. We assume that such a pair $\overline{\gamma}_H, \overline{\gamma}_G$ exists; otherwise all the following factors are to be zero.

Definition. For all strongly G-regular $\gamma_H \in H(F)$,

$$\Delta_{\mathbf{A}}(\gamma_H, \gamma_G) = d(\overline{\gamma}_H, \overline{\gamma}_G)/d(\gamma_H, \gamma_G)$$

if γ_H is an adelic image of $\gamma_G \in G(\mathbf{A})$ and $\Delta_{\mathbf{A}}(\gamma_H, \gamma_G) = 0$ otherwise.

By Lemma 6.4.B, $\Delta_{\mathbf{A}}(\gamma_H, \gamma_G)$ is independent of: (i) the pair $\overline{\gamma}_H, \overline{\gamma}_G$; (ii) γ_H within its stable conjugacy class; (iii) γ_G within its $G(\mathbf{A})$-conjugacy class. Further,

$$\Delta_{\mathbf{A}}(\gamma_H, \gamma_G) = 1$$

if γ_H is an adelic image of γ_G and the $G(\mathbf{A})$-conjugacy class of γ_G meets $G(F)$.

(6.4) *Product Formulas*

To specify the local factors of (3.7) we use a pair $\overline{\gamma}_H, \overline{\gamma}_G$ of F-rational elements, as for the adelic factor. At almost all places v we set $\Delta^{(v)}(\overline{\gamma}_H, \overline{\gamma}_G) = 1$. At the remaining places $\Delta^{(v)}(\overline{\gamma}_H, \overline{\gamma}_G)$ is arbitrary except for the requirement that

$$\prod_v \Delta^{(v)}(\overline{\gamma}_H, \overline{\gamma}_G) = 1.$$

Then as in (3.7), but with the superscript (v) inserted, we set

$$\Delta^{(v)}(\gamma_H, \gamma_{G,v}) = \Delta^{(v)}(\overline{\gamma}_H, \overline{\gamma}_{G,v})\Delta^{(v)}(\gamma_H, \gamma_{G,v}; \overline{\gamma}_H, \overline{\gamma}_G)$$

for all strongly G-regular γ_H in $H(F)$. The relative factor $\Delta^{(v)}(\gamma_H, \gamma_{G,v}; \overline{\gamma}_H, \overline{\gamma}_G)$ is the product of

$$\frac{\Delta_I^{(v)}(\gamma_H, \gamma_{G,v})}{\Delta_I^{(v)}(\overline{\gamma}_H, \overline{\gamma}_G)} \cdot \frac{\Delta_{II}^{(v)}(\gamma_H, \gamma_{G,v})}{\Delta_{II}^{(v)}(\overline{\gamma}_H, \overline{\gamma}_G)} \cdot \frac{\Delta_2^{(v)}(\gamma_H, \gamma_{G,v})}{\Delta_2^{(v)}(\overline{\gamma}_H, \overline{\gamma}_G)}$$

and

$$\frac{\Delta_{IV}^{(v)}(\gamma_H, \gamma_{G,v})}{\Delta_{IV}^{(v)}(\overline{\gamma}_H, \overline{\gamma}_G)} \cdot \Delta_1(\gamma_H, \gamma_{G,v}; \overline{\gamma}_H, \overline{\gamma}_G).$$

The various terms are defined using any admissible embeddings over F_v, and a-data, χ-data for G as group over F_v. We shall use embeddings over F and global a- and χ-data [see (2.2), (2.5)] in order to obtain product formulas for the individual terms as well as for Δ.

Theorem 6.4.A. (i) *For almost all v each of* $\Delta_I^{(v)}(\gamma_H, \gamma_{G,v})$, $\Delta_{II}^{(v)}(\gamma_H, \gamma_{G,v})$, $\Delta_2^{(v)}(\gamma_H, \gamma_{G,v})$ *and* $\Delta_{IV}(\gamma_H, \gamma_{G,v})$ *equals* 1.

(ii) $\prod_v \Delta_I^{(v)}(\gamma_H, \gamma_{G,v}) = 1$, *and similarly for* $\Delta_{II}^{(v)}, \Delta_2^{(v)}$, *and* $\Delta_{IV}^{(v)}$.

In the last section we showed that

$$\Delta_I^{(v)}(\gamma_H, \gamma_{G,v}; \overline{\gamma}_H, \overline{\gamma}_G) = 1$$

for almost all v (Lemma 6.3.A) and that

$$\prod_v \Delta_I^{(v)}(\gamma_H, \gamma_{G,v}; \overline{\gamma}_H, \overline{\gamma}_G) = \Delta_{\mathbf{A}}(\gamma_H, \gamma_G)$$

[(6.3.1) and (6.3.2)].

Corollary 6.4.B.

(i) $\Delta_{(v)}(\gamma_H, \gamma_{G,v}) = 1$ *for almost all* v *and*

(ii) $\prod_v \Delta^{(v)}(\gamma_H, \gamma_{G,v}) = \Delta_{\mathbf{A}}(\gamma_H, \gamma_G)$.

This product formula contains the Global Hypothesis of [L2, p. 149] because $\Delta_{\mathbf{A}}(\gamma_H, \gamma_G)$ is the term $\kappa(\epsilon(D))$ of [L2]. To see this we translate our terminology into that of diagrams, as at the end of (4.2). The formula for $\epsilon(D)$ on p. 137 of [L2] determines an element of $H^{-1}(\Gamma, X_*(U))$ which under global Tate-Nakayama duality coincides with our μ of (6.3). Then $\kappa(\epsilon(D))$ is $\langle \mu, \mathbf{s}_U \rangle$ which is the same as $\Delta_{\mathbf{A}}(\gamma_H, \gamma_G)$.

Proof of Theorem. (Δ_I) By definition,

$$\Delta_I^{(v)}(\gamma_H, \gamma_{G,v}) = \langle \lambda_v(T_{\mathrm{sc}}), \mathbf{s}_{T,v} \rangle$$

[see (3.2)]. Recall from (2.3.5) that the global invariant $\lambda(T_{\mathrm{sc}}) \in H^1(\Gamma, T(L))$ is defined and that $\lambda_v(T_{\mathrm{sc}})$ is the image of $\lambda(T_{\mathrm{sc}})$ under $H^1(\Gamma, T(L)) \to H^1(\Gamma_v, T(L_v))$. Thus for almost all v we have $\lambda_v(T_{\mathrm{sc}}) = 1$ and then $\Delta_I^{(v)}(\gamma_H, \gamma_{G,v}) = 1$. Further, the image of $\Sigma_v \lambda_v(T_{\mathrm{sc}})$in $H^1(\Gamma, T_{\mathrm{sc}}(L)\backslash T_{\mathrm{sc}}(\mathbf{A}_L))$ is trivial and so

$$\prod_v \Delta_{(I)}^{(v)}(\gamma_H, \gamma_{G,v}) = \prod_v \langle \lambda_v, \mathbf{s}_{T,v} \rangle = \langle \text{Image } \Sigma_v \lambda_v, \mathbf{s}_T \rangle = 1.$$

(Δ_{II}) In (2.6.5) we attached local χ-data $\{\chi_\alpha^{(v)}\}$ to global data $\{\chi_\alpha\}$. The character $\chi_\alpha^{(v)}$ is defined on $F_{v,\alpha}$, the fixed field in L_v of the stabilizer of α in Γ_v, and

$$\Delta_{II}^{(v)}(\gamma_H, \gamma_{G,v}) = \prod_\alpha \chi_\alpha^{(v)} \left(\frac{\alpha(\gamma) - 1}{a_\alpha} \right),$$

where the product is over representatives α for the orbits of Γ_v in the roots of T which lie outside H. Note that $\delta = \frac{\alpha(\gamma)-1}{a_\alpha}$ lies in F_α, the fixed field in L of the stabilizer of α in Γ. On the other hand we may write χ_α as $\prod_v \chi_{\alpha,v}$ where $\chi_{\alpha,v}$ is a character on $(F_\alpha \otimes F_v)^\times$. Then we claim that

$$\Delta_{II}^{(v)}(\gamma_H, \gamma_{G,v}) = \prod\nolimits_\alpha' \chi_{\alpha,v}(\delta),$$

where the product is now over representatives α for the orbits of Γ in the set of roots outside H. Then

$$\Delta_{II}^{(v)}(\gamma_H, \gamma_{G,v}) = 1$$

for almost all v and

$$\prod_v \Delta_{II}^{(v)}(\gamma_H, \gamma_{G,v}) = \prod_v \prod\nolimits_\alpha' \chi_{\alpha,v}(\delta) = \prod\nolimits_\alpha' \prod_v \chi_{\alpha,v}(\delta) = \prod\nolimits_\alpha' \chi_\alpha(\delta) = 1.$$

To prove the claim we fix a root α outside H and choose representatives $\sigma_1,\ldots,\sigma_n$ for $\Gamma_v\backslash\Gamma/\Gamma_\alpha$, where Γ_α is the stabilizer of α in Γ. Then $\sigma_1\alpha,\ldots,\sigma_n\alpha$ are representatives for the Γ_v-orbits in the Γ-orbit of α. Thus the contribution to $\Delta_{II}^{(v)}(\gamma_H,\gamma_{G,v})$ from these orbits is $\prod_i \chi^{(v)}_{\sigma_i\alpha}(\sigma_i\delta)$. But $F_\alpha\otimes F_v=\prod_i F_{\alpha,\sigma_i^{-1}v}=\prod_i \sigma_i^{-1}(F_{v,\sigma_i\alpha})$. From $\chi_\alpha=\chi_{\sigma_i\alpha}\cdot\sigma_i$ we conclude that if $\chi_{\alpha,v}=\prod_i\chi_i$ then $\chi_i=\chi^{(v)}_{\sigma_i\alpha}\cdot\sigma_i$, and the claim is proved.

(Δ_2) By definition,

$$\Delta_2^{(v)}(\gamma_H,\gamma_{G,v})=\langle\mathbf{a}_v,\gamma\rangle,$$

where $\mathbf{a}_v$ is the element $\mathbf{a}$ of $H^1(W_v,\widehat{T})$ constructed in (3.5). Because we have global χ-data and $T_H\to T$ is defined over F we may similarly construct $\mathbf{a}\in H^1(W,\widehat{T})$. By (2.6.5) we have that $\mathbf{a}_v$ is the image of $\mathbf{a}$ under $H^1(W,\widehat{T})\to H^1(W_v,\widehat{T})$. Thus $\Delta_2(\gamma_H,\gamma_{G,v})=1$ for almost all v and $\prod_v\Delta_2^{(v)}(\gamma_H,\gamma_{G,v})=\prod_v\langle\mathbf{a}_v,\gamma\rangle=\langle\mathbf{a},\gamma\rangle=1$ since $\gamma\in T(F)$.

The assertions of the theorem are immediate for Δ_{IV}. This completes the proof.

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