On Artin's L-Functions*

by

R.P. Langlands

The nonabelian Artin L-functions and their generalizations by Weil are known to be meromorphic in the whole complex plane and to satisfy a functional equation similar to that of the zeta function but little is known about their poles. It seems to be expected that the L-functions associated to irreducible representations of the Galois group or another closely related group, the Weil group, will be entire. One way that has been suggested to show this is to show that the Artin L-functions are equal to L-functions associated to automorphic forms. This is a more difficult problem than we can consider at present.

In certain cases the converse question is much easier. It is possible to show that if the L-functions associated to irreducible two-dimensional representations of the Weil group are all entire then these L-functions are equal to certain L-functions associated to automorphic forms on $GL(2)$. Before formulating this result precisely we review Weil's generalization of Artin's L-functions.

A global field will be just an algebraic number field of finite degree over the rationals or a function field in one variable over a finite field. A local field is the completion of a global field at some place. If F is a local field let C_F be the multiplicative group of F and if F is a global field let C_F be the idèle class group of F. If K is a finite Galois extension of F the Weil group $W_{K/F}$ is an extension of $\mathfrak{G}(K/F)$, the Galois group of K/F, by C_K. Thus there is an exact sequence

$$1 \to C_K \to W_{K/F} \to \mathfrak{G}(K/F) \to 1.$$

If L/F is also Galois and L contains K there is a continuous homomorphism $\tau_{L/F,K/F}$ of $W_{L/F}$ onto $W_{K/F}$. It is determined up to an inner automorphism of $W_{K/F}$ by an element of C_K. In particular $W_{F/F} = C_F$ and the kernel of $\tau_{K/F,F/F}$ is the commutator subgroup of $W_{K/F}$. Also if $F \subseteq E \subseteq K$ we may regard $W_{K/E}$ as a subgroup of $W_{K/F}$. If F is global and v is a place of F we also denote by v any extension of v to K. There is a homomorphism α_v of W_{K_v/F_v} into $W_{K/F}$ which is determined up to an inner automorphism by an element of C_K.

A representation σ of $W_{K/F}$ is a continuous homomorphism of $W_{K/F}$ into the group of invertible linear transformations of a finite-dimensional complex vector space such that $\sigma(w)$ is diagonalizable

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for all w in $W_{K/F}$. If $F \subseteq K \subseteq L$ then $\sigma \circ \tau_{L/F,K/F}$ is a representation of $W_{L/F}$ which we need not distinguish from σ. In particular, one-dimensional representations of $W_{K/F}$ may be identified with quasi-characters of C_F, that is, continuous homomorphisms of C_F into the multiplicative group of complex numbers. If ω is such a quasi-character and σ a representation of $W_{K/F}$ then $\omega \otimes \sigma$ has the same dimension as σ. If $F \subseteq E \subseteq K$ and ρ is a representation of $W_{K/E}$ on X let Y be the space of functions ϕ on $W_{K/F}$ with values in X such that

$$\phi(uw) = \rho(u)\phi(w)$$

for all u in $W_{K/E}$. If ν is in $W_{K/F}$ we set

$$\sigma(v)\phi(w) = \phi(wv);$$

σ is a representation of $W_{K/F}$ on Y and we write

$$\sigma = \text{Ind}(W_{K/F}, W_{K/E}, \rho).$$

If F is a global field and σ is a representation of $W_{K/F}$ then, for all places $v, \sigma_v = \sigma \circ \alpha_v$ is a representation of W_{K_v/F_v} whose class depends only on that of σ.

Now take F to be a local field. Suppose ω is a quasi-character of C_F. If F is nonarchimedean and ω is ramified, we set $L(s,\omega) = 1$ but if ω is unramified we set

$$L(s,\omega) = \frac{1}{1 - \omega(\varpi)|\varpi|^s};$$

ϖ is a uniformizing parameter for F. If F is real and $\omega(x) = (\text{sgn}\ x)^m |x|^r$ with m equal to 0 or 1 we set

$$L(s,\omega) = \pi^{-\frac{1}{2}(s+r+m)}\Gamma\left(\frac{s+r+m}{2}\right);$$

if F is complex and

$$\omega(z) = (z\bar{z})^{r-(m+n)/2} z^m \bar{z}^n$$

with $m + n \geq 0, mn = 0$ then

$$L(s,\omega) = 2(2\pi)^{-(s+r+(m+n)/2)}\Gamma\left(s + r + \frac{m+n}{2}\right).$$

The local L-functions $L(s,\sigma)$ associated to representations of the Weil groups $W_{K/F}$ are characterized by the following four properties.

(i) $L(s,\sigma)$ depends only on the class of σ.

(ii) If σ is of dimension 1 and thus associated to a quasi-character ω of C_F then

$$L(s, \sigma) = L(s, \omega).$$

(iii)

$$L(s, \sigma_1 \oplus \sigma_2) = L(s, \sigma_1) L(s, \sigma_2).$$

(iv) If $F \subseteq E \subseteq K$, ρ is a representation of $W_{K/E}$, and

$$\sigma = \text{Ind}(W_{K/F}, W_{K/E}, \rho),$$

then

$$L(s, \sigma) = L(s, \rho).$$

Now suppose $\psi = \psi_F$ is a nontrivial additive character of F. If E is a finite separable extension of F let

$$\psi_{E/F}(x) = \psi_F(Tr_{E/F} x).$$

Suppose E is any local field and ψ_E is a nontrivial additive character of E. If E is nonarchimedean and ω_E is a quasi-character of C_E let $n = n(\psi_E)$ be the largest integer such that ψ_E is trivial on $\mathfrak{P}_E^{-n}$ and let $m = m(\omega_E)$ be the order of the conductor of ω_E. If U_E is the group of units in the ring of integers of E and γ is any element of F with order $m + n$ let

$$\Delta(\omega_E, \psi_E) = \omega_E(\gamma) \frac{\int_{U_E} \psi_E\left(\frac{\alpha}{\gamma}\right) \omega_E^{-1}(\alpha) d\alpha}{\left|\int_{U_E} \psi_E\left(\frac{\alpha}{\gamma}\right) \omega_E^{-1}(\alpha) d\alpha\right|}.$$

If E is real and $\psi_E(z) = e^{2\pi i u x}$ while $\omega_E(x) = (\text{sgn } x)^m |x|^r$ then

$$\Delta(\omega_E, \psi_E) = (i \text{ sgn } u)^m |u|^r.$$

If E is complex and $\psi_E(z) = e^{4\pi i \text{Re}(wz)}$ while

$$\omega_E(z) = (z\bar{z})^{r-(m+n)/2} z^m \bar{z}^n$$

then

$$\Delta(\omega_E, \psi_E) = i^{m+n} \omega_E(w).$$

Theorem 1. *Suppose F is a local field and ψ_F is a nontrivial additive character of F. It is possible in exactly one way to associate to every finite separable extension E of F a complex number*

$\lambda(E/F,\psi_F)$ and to every representation σ of any Weil group $W_{K/E}$ a complex number $\epsilon(\sigma,\psi_{E/F})$ so that the following conditions are satisfied.

(i) *$\epsilon(\sigma,\psi_{E/F})$ depends only on the class of σ.*

(ii) *If σ is one-dimensional and corresponds to the quasi-character ω_E then*

$$\epsilon(\sigma,\psi_{E/F})=\Delta(\omega_E,\psi_{E/F}).$$

(iii)

$$\epsilon(\sigma_1\oplus\sigma_2,\psi_{E/F})=\epsilon(\sigma_1,\psi_{E/F})\epsilon(\sigma_2,\psi_{E/F}).$$

(iv) *If*

$$\sigma=\ \mathit{Ind}(W_{K/F},W_{K/E},\rho),$$

then

$$\epsilon(\sigma,\psi_F)=\epsilon(\rho,\psi_{E/F})\lambda(E/F,\psi_E)^{\dim\rho}.$$

If s is a complex number and $|a|_F$ is the normalized absolute value on F set $\alpha_F^s(a)=|a|_F^s$ and set

$$\epsilon(s,\sigma,\psi_F)=\epsilon(\alpha^{s-\frac{1}{2}}\otimes\sigma,\psi_F).$$

If F is a global field and σ is a representation of $W_{K/F}$ the global L-function $L(s,\sigma)$ is defined by

$$L(s,\sigma)=\prod_v L(s,\sigma_v).$$

The product is taken over all places including the archimedean ones. It converges in a right half-plane. $L(s,\sigma)$ can be analytically continued to the whole plane as a meromorphic function. If ψ is a nontrivial character of $F\backslash A$, the quotient of the ring of adèles by F, then, for each place v, the restriction ψ_v of ψ to F_v is nontrivial. For almost all v the factor $\epsilon(s,\sigma_v,\psi_v)$ is 1 and

$$\epsilon(s,\sigma)=\prod_v \epsilon(s,\sigma_v,\psi_v)$$

does not depend on the choice of ψ.

Theorem 2. *If $\tilde{\sigma}$ is the contragradient of σ the functional equation*

$$L(s,\sigma) = \epsilon(s,\sigma)L(1-s,\tilde{\sigma})$$

is satisfied.

The proofs of these two theorems may be found in [2]. Now we have to say something about the L-functions associated to automorphic forms on $GL(2)$.

If F is a local field and π is an irreducible representation of the locally compact group $GL(2,F)$ we can introduce a local L-function $L(s,\pi)$. Given a nontrivial additive character ψ of F we can also introduce a factor $\epsilon(s,\pi,\psi)$. If ω is a quasi-character of C_F then $\omega \otimes \pi$ will be the representation $g \to \omega(\det g)\pi(g)$.

If F is a global field and η is a quasi-character of C_F then $A_0(\eta)$ will be the space of all measurable functions ϕ on $GL(2,F)\backslash GL(2,\mathbf{A})$ satisfying:

(i) For every idèle a

$$\phi\left(\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} g\right) = \eta(a)\phi(g);$$

(ii) If $Z_{\mathbf{A}}$ is the group of scalar matrices in $GL(2,\mathbf{A})$

$$\int_{Z_{\mathbf{A}}GL(2,F)\backslash GL(2,\mathbf{A})} |\phi(g)|^2|\eta(\det g)|^{-1}dg$$

is finite;

(iii) For almost all g in $GL(2,A)$

$$\int_{F\backslash\mathbf{A}} \phi\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g\right) g\Big)\, dx = 0.$$

$GL(2,\mathbf{A})$ acts on $A_0(\eta)$ by right translations. Under this action $A_0(\eta)$ breaks up into the direct sum of mutually orthogonal, invariant, irreducible subspaces. Let π be the representation of $GL(2,A)$ on one of these subspaces. One can decompose π into a tensor product $\otimes_v \pi_v$ where π_v is an irreducible representation of $GL(2,F_v)$. The product

$$L(s,\pi) = \prod_v L(s,\pi_v)$$

converges in a half plane and $L(s,\pi)$ can be analytically continued to an entire function. If ψ is a nontrivial character of $F\backslash A$ and

$$\epsilon(s,\pi) = \prod_v \epsilon(s,\pi_v,\psi_v)$$

then

$$L(s,\pi)=\epsilon(s,\pi)L(1-s,\tilde{\pi}).$$

Now suppose F is a local field and σ is a two-dimensional representation of $W_{K/F}$. Then $\det\sigma$ is a one-dimensional representation of $W_{K/F}$ and may therefore be regarded as a quasi-character of C_F. There is at most one irreducible representation $\pi=\pi(\sigma)$ of $GL(2,F)$, satisfying the following conditions.

(i) For all a in C_F

$$\pi\left(\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}\right)=\det\sigma(a)I$$

if I is the identity.

(ii) For every quasi-character ω of C_F

$$L(s,\omega\otimes\pi)=L(s,\omega\otimes\sigma),$$

and

$$L(s,\omega\otimes\tilde{\pi})=L(s,\omega\otimes\tilde{\sigma}).$$

(iii) For one, and hence for every, nontrivial character ψ of F

$$\epsilon(s,\omega\otimes\pi,\psi)=\epsilon(s,\omega\otimes\sigma,\psi)$$

for all ω.

Theorem 3. *Suppose F is a global field and σ is a two-dimensional representation of $W_{K/F}$. Let $\eta=\det\sigma$. Suppose that for every quasi-character ω of C_F both $L(s,\omega\otimes\sigma)$ and $L(s,\omega^{-1}\otimes\tilde{\sigma})$ are entire functions which are bounded in vertical strips of finite width. Then $\pi_v=\pi(\sigma_v)$ exists for each v and the representation $\otimes_v\pi_v$ of $GL(2,\mathbf{A})$ occurs in the representation of $GL(2,\mathbf{A})$ on $A_0(\eta)$.*

This theorem is proved in [1], written in collaboration with H. Jacquet.

References

1. Jacquet, H. and R.P. Langlands, *Automorphic forms on $GL(2)$*, Lecture Notes in Mathematics, No. 114, Springer-Verlag (1970).

2. Langlands, R.P., *On the functional equation of the Artin L-functions*, (in preparation).

Yale University