Automorphic Forms on GL(2) *

In [3] Jacquet and I investigated the standard theory of automorphic forms from the point of view of group representations. I would like on this occasion not only to indicate the results we obtained but also to justify our point of view.

For us it is imperative not to consider functions on the upper half plane but rather to consider functions on $GL(2, \mathbf{Q}) \backslash GL(2, \mathbf{A}(\mathbf{Q}))$ where $\mathbf{A}(\mathbf{Q})$ is the adéle ring of $\mathbf{Q}$. We also replace $\mathbf{Q}$ by an arbitrary number field or function field (in one variable over a finite field) F. One can introduce [3] a space of functions, called automorphic forms, and the notion that an irreducible representation π of $GL(2, \mathbf{A}(F))$ is a constituent of the space of automorphic forms on $GL(2, F) \backslash GL(2, \mathbf{A}(F))$.

Such a representation can in a certain sense be written as a tensor product

$$\pi = \otimes_v \pi_v$$

where the product is taken over all valuations of the field and π_v is an irreducible represenation of $GL(2, F_v)$. F_v is the completion of F at v. Such a π_v has associated to it a local zeta-function $L(s, \pi_v)$ which can be expressed in terms of Γ-functions if v is archimedean and otherwise is of the form

$$\frac{1}{(1 - \alpha|\widetilde{\omega}_v|^s)(1 - \beta|\widetilde{\omega}_v|^s)}$$

if $\widetilde{\omega}_v$ is a uniformizing parameter for F_v. If ψ is a non-trivial character of $F \backslash \mathbf{A}(F)$ and ψ_v is its restriction to F_v there are also factors $\epsilon(s, \pi_v, \psi_v)$ which for the given collection of π_v are almost all 1. The functions

$$L(s, \pi) = \Pi_v L(s, \pi_v)$$

and, if $\widetilde{\pi}_v$ is the representation contragredient to π_v,

$$L(s, \widetilde{\pi}) = \Pi_v L(s, \widetilde{\pi}_v)$$

are defined by the products on the right for s in a half-plane and can be analytically continued as meromorphic functions to all of the complex plane. If

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* Appeared in *Proceedings of the International Congress of Mathematicians*, Nice (1970)

$$\epsilon(s,\pi) = \Pi_v \epsilon(s,\pi_v,\psi_v)$$

then

$$L(s,\pi) = \epsilon(s,\pi)L(1-s,\widetilde{\pi})$$

This is of course nothing but the functional equation of Hecke and of Maass except that the ground field is now more general. The converse theorem can take a number of forms. I do not give them here (cf [3]) but do mention that they involve the basic idea of [6].

The converse theorems can be used to justify some suggestions of mine [4] as well as some of Weil's [6] to which they are closely related. Assume that Weil's generalizations of the Artin L-functions are for irreducible two-dimensional representations of the Weil group entire functions. Then locally one has a map

$$\sigma_v \to \pi(\sigma_v)$$

from the two dimensional representations of the Weil group of F_v to the irreducible representations of $GL(2,F_v)$ and if σ is a representation of the Weil group of the global field F and σ_v its restriction to the Weil group of F_v the representation

$$\pi(\sigma) = \otimes_v \pi(\sigma_v)$$

is a constituent of the representation of $GL(2,\mathbf{A}(F))$ on the space of automorphic forms. Moreover the Artin L-function $L(s,\sigma)$ is equal to $L(s,\pi(\sigma))$. Only for function fields does this assertion lead to a real theorem.

Weil's suggestions have, I understand, also been verified for function fields [7]. It is possible, by stressing their local aspects, to refine these suggestions slightly. Although these refined suggestions have, so far as I know, not been verified I would like to mention them. If C is an elliptic curve over F, say of characteristic 0, then, as in [4], we can associate to each F_v a representation $\pi(C/F_v)$ of $GL(2,F_v)$ which depends only on C as a curve over F_v. The refined form of Weil's suggestion is that

$$\pi(C) = \otimes_v \pi(C/F_v)$$

is a constituent of the space of automorphic forms. $L(s,\pi(C))$ would then be, apart perhaps from a translation, the zeta-function $L(s,C)$ of the curve. Since $\pi(C)$ is its own contragredient we would have

$$L(s,C)=\epsilon(s,\pi(C))L(1-s,C)$$

and the factor

$$\epsilon(s,\pi(C))=\Pi_v\epsilon(s,\pi_v,\psi_v)$$

could be computed in terms of local properties of the curve without reference to the theory of automorphic forms. For those elliptic curves C over $\mathbf{Q}$ which sit in the Jacobians of the curves associated to elliptic modular functions one knows that a π occurs in the space of automorphic forms whose local factors are equal to those of $\pi(C)$ at almost all places. The assertion that they are equal everywhere is an interesting assertion about, among other things, the j-invariants of such curves. Some examples can be found in [2]. I understand that Deligne has obtained a reasonably general result along these lines. Casselman [1] has established results of a similar nature.

If G' is the multiplicative group of a quaternion algebra over F and π' is a represenation of $G'_{\mathbf{A}(F)}$ occurring in the space of automorphic forms on $G'_F\backslash G'_{\mathbf{A}(F)}$ one can form a zeta-function

$$L(s,\pi')=\Pi_v L(s,\pi'_v)$$

with the usual properties. Here again π' can be written in a certain sense as

$$\pi'=\otimes_v\pi'_v.$$

It is possible to define locally maps

$$\pi'_v\to\pi(\pi'_v)$$

from the representations of G'_{F_v} to those of $GL(2,F_v)$ so that globally

$$\pi(\pi')=\otimes_v\pi(\pi'_v)$$

is a constituent of the space of automorphic forms on $GL(2,F)\backslash GL(2,\mathbf{A}(F))$ if π' is a constituent of the space of automorphic forms on $G'_F\backslash G'_{\mathbf{A}(F)}$. Again

$$L(s,\pi') = L(s,\pi(\pi')).$$

If the field F has characteristic 0 the relations, mostly hypothetical, between the functions $L(s,\sigma)$, $L(s,C)$, $L(s,\pi')$ and the functions $L(s,\pi)$ do not, unlike the usual identities between L-functions, imply elementary number-theoretical statements because the local factors in the Euler products defining $L(s,\pi)$ are determined transcendentally. The corresponding factors for $L(s,\sigma)$, $L(s,C)$ and, at least when the quaternion algebra does not split at any archimedean prime, $L(s,\pi')$ are however determined in an elementary way. Thus relations between the functions $L(s,\sigma)$, $L(s,C)$ on one hand and $L(s,\pi')$ on the other are of some interest. Scattered instances of such relations can be found in the literature. To obtain a general theorem along these lines one needs a criterion for deciding when a representation π of $GL(2,\mathbf{A}(F))$ occurring in the space of automorphic forms is the representation corresponding to a representation π' occurring in the space of automorphic forms on $G'_F\backslash G'_{\mathbf{A}(F)}$. Jacquet and I sketched a proof that that is so precisely when the local factors π_v of π belong to the discrete series for all valuations at which the quaternion algebra does not split. This criterion is easily applied to the representations $\pi(\sigma)$ and $\pi(C)$.

References

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