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Dear Godement,

I do not know if this letter will serve as a report on Jacquet's thesis. If a shorter statement or one of a different nature is necessary I would be willing to try to write it. In any case I wanted to write down as well as I could in the time available my thoughts on reading the thesis. In particular there are a number of problems suggested by the results of Jacquet and I would like to know if he has done anything with them.

I would like to see everything done in terms of general reductive groups for it would probably be useful to solve the technical problems which would arise. Consider first the question of convergence. It appears that the local reduction theory, as developed by Bruhat and Tits, is now in sufficiently good shape that the method of / 2 in the thesis can be used to reduce the question of convergence to the rank-one case. Let G be a reductive group over K with minimal parabolic subgroup P for which, when $K = \mathbb{Q}_p$, conjecture I of Bruhat's Boulder paper is satisfied. (It will be convenient to use his notation.) If $\bar{U}$ is the unipotent radical of the parabolic subgroup opposed to P one has to prove that

$$\int_{\bar{U}_K} L(g\bar{u}, \lambda + \rho)\, d\bar{u}$$

converges when $Re(\lambda, \alpha) > 0$ for the simple roots of A. Here $L(g, \lambda)$ is the obvious generalization of the corresponding function in Jacquet. Harish-Chandra has shown in his first paper on spherical functions that this is so when $K = \mathbb{R}$. There are two parts to his proof. He first studies the asymptotic behaviour of spherical functions; then he shows that the convergence of the integral is implied by the nature of the asymptotic behaviour of a particular spherical function. Although there would be some difficulties it appears that the second step can be carried through without major changes when $K = \mathbb{Q}_p$. The first step is perhaps rather easier when $K = \mathbb{Q}_p$.

Let H be the algebra of compactly supported functions on G bi-invariant under K. Let $f \mapsto \widetilde{f}$ be the homomorphism of H into the group algebra of A_K/A_O defined by

$$\widetilde{f}(a) = L(a, -\rho) \int_{U_K} f(au)\, du\ .$$

Presumably Satake's results are valid so the image of H is the set of elements in the group algebra which are invariant under the Weyl group. Suppose φ is bi-invariant under K and

$$\lambda(f)\varphi(g) = \int_{G_K} f(h^{-1})\varphi(hg)\, dg \equiv \chi(f)\varphi(g)$$

where $f \mapsto \chi(f)$ is a homomorphism of H into $\mathbb{C}$. One need only study the asymptotic behaviour of φ on the 'positive Weyl chamber' in A_K. (To avoid inessential complications assume that $Z_K = A_K Z_O$.) If ψ_1 and ψ_2 are two functions on A_K/A_O then ψ_1 and ψ_2 will be said to be equivalent if there is a constant $c > 0$ such that

$\psi_1(a) = \psi_2(a)$ whenever $\langle \alpha, a \rangle \geq c$ for all simple roots α. Let W be the vector space formed by these equivalence classes. The group algebra of A_K/A_O acts on W. Let $\psi(a) = L(a, -\rho)\varphi(a)$ and let Ψ be the class of ψ in W. Once one shows that Ψ is annihilated by an ideal of finite codimension in the group algebra one sees that there is a closed expression for ψ which is valid if $\langle \alpha, a \rangle$ is sufficiently large for all simple roots. It is enough to show that

$$\lambda(\widetilde{f})\Psi = \chi(f)\Psi$$

if f is in H. Choose a compact set ω in U_K such that the support of f is contained in $\omega U_K K$. If $a^{-1}\omega a \subseteq U_O$ then

$$\begin{aligned}
\chi(f)\varphi(a) &= \int_K \int_{A_K} \int_{U_K} f(u^{-1}b^{-1}k^{-1})\varphi(kbua)L(b, -2\rho)\, du\, db\, dk \\
&= \int_{A_K} \int_\omega f(u^{-1}b)\varphi(kba)L(b, -2\rho)\, du\, db \\
&= \int_{A_K} \varphi(ba) \int \left\{ \int_{U_K} f(b^{-1}u)\, du \right\} db \\
&= \int_{A_K} L(b, -\rho)\varphi(ba)\widetilde{f}(b^{-1})\, db\ .
\end{aligned}$$

Of course the most interesting feature of Jacquet's thesis is the functional equation. As I mentioned to you before I believe there is a uniqueness theorem behind it. Consider first the case that $K = \mathbb{R}$. For simplicity take $G_{\mathbb{R}}$ connected. Let χ be a generic character of U_K and let $L(\chi)$ be the space of all K-finite (there are two different K's here) functions on $G_{\mathbb{R}}$ satisfying

(i) $\varphi(gu) = \overline{\chi(u)}\varphi(g)$ for all u in U_K

(ii) There are vectors $\lambda_1, \ldots, \lambda_n$ and constants $a_1, \ldots, a_n$ such that

$$|\varphi(ka)| = \sum_{i=1}^{n} a_i |\langle \lambda_i, a \rangle|\ .$$

Do most, if not all, quasi-simple irreducible representations of the universal enveloping algebra occur at most once in $L(\chi)$? I have no idea at the moment how to prove such a uniqueness theorem when the rank is greater than one. Such a proof would be of use for studying similar questions in the harmonic analysis on G and $\Gamma\backslash G$. However in the rank-one case it might be possible to prove it by using an analogue of what Harish-Chandra calls the Maass-Selberg relations.

Suppose then the rank of G is one. Let KAN be the Iwasawa decomposition and let M be the centralizer of A in K. Let $\mathfrak{Z}$ be the centre of the universal enveloping algebra $\mathfrak{A}$ of G and $\mathfrak{Z}_1$ the centre of the universal enveloping algebra of M. The normalizer of A acts on the homomorphisms of $\mathfrak{Z}_1$ into $\mathbb{C}$. Let $\zeta_1, \ldots, \zeta_\ell$ be the orbit of the homomorphism ζ and let $L(s, \zeta)$ be the space of K-finite functions on $G_{\mathbb{R}}/U_{\mathbb{R}}$ which satisfy

(i) $\varphi(ga) = < a, -s\alpha - \rho > \varphi(g)$ for $a \in A_{\mathbb{R}}$ (α is the simple root)

(ii) $\prod_{i=1}^{\ell} \big(\rho(Z) - \zeta_i(Z)\big) = 0$ for $Z \in Z_1$.

Let $\pi_{s,\zeta}$ be the representation of $\mathfrak{A}$ on $L(s,\zeta)$. All representations of $\mathfrak{A}$ which are relevant to us are contained in some $\pi_{s,\zeta}$. If $Z \in \mathfrak{Z}$ then $\pi_{s,\zeta}(Z)$ is a scalar.

Fix an irreducible representation π of K. Let $L(\chi,s,\zeta,\pi)$ be the set of all functions in $L(\chi)$ which transform under K according to π and satisfy

$$(A) \qquad \lambda(Z)\varphi = \pi_{s,\zeta}(Z)\varphi \ .$$

For such φ there is a constant vector Ψ and a vector function $\Phi(t)$ defined for $t>0$ such that

$$\varphi(ka) = {}^t\Phi(t)\pi(k^{-1})\Psi$$

if $\langle a,\alpha\rangle = t^{-1}$, α being the simple root of A. Let $D = td/dt$. Using Harish-Chandra's results on the structure of the centre of the universal enveloping algebra one should be able to show that equation (A) is equivalent to an equation of the form

$$\sum_{k=0}^{n(Z)} A_k(t,Z)D^k\Phi = \pi_{s,\zeta}(Z)\Phi$$

where $A_k(t,Z)$ is a polynomial in t. If one replaces χ by the trivial character and repeats the process one obtains the equation

$$\sum_{k=0}^{n(Z)} A_k(0,Z)D^k\Phi = \pi_{s,\zeta}(Z)\Phi \ .$$

Moreover if D is the Casimir operator then $n(D)=2$ and $A_2(0,D)$ is a non-zero scalar. Thus one can presumably apply the method of Frobenius to show that, except for certain exceptional values of s,

$$\Phi(t) = t^{+s+a_0}(\Phi_1 + O(t)) + t^{-s+a_0}(\Phi_2(t) + O(t))$$

if $\rho = a_0\alpha$.

One should also be able to show by using the methods of the theory of Eisenstein series or of asymptotic expansions of ordinary differential equations that if M is real then $\Phi(t) = O(t^M)$ as $t \to \infty$. Suppose φ lies in $L(\chi,s,\zeta,\pi)$ and φ' lies in $L(\widetilde{\chi},s',\zeta',\widetilde{\pi})$. $\widetilde{\pi}$ is the contragredient of π and $\widetilde{\chi}$ is the contragredient of χ. If B_x is the image in $N\backslash G$ of $\{kan \mid \langle a,\alpha\rangle < -x\}$ then

$$\{\pi_{s\zeta}(D) - \pi_{s',\zeta'}(D)\} \int_{B_x} \{\lambda(D)\varphi\varphi' - \varphi\lambda(D)\varphi'\}\, d\bar{g}$$

is equal to

$$\text{const} \times t^{-2a_0} \left\{ {}^t\left(t\frac{d}{dt}\Phi(t)\right)\Phi'(t) - {}^t\Phi(t)\left(\overline{t\frac{d}{dt}\Phi'(t)}\right)\right\}_{t=x} \times {}^t\Psi\bar{\Psi}' \ .$$

In particular if $s=s'$, $\zeta=\zeta'$ the expression

$$t^{-2a_0}\left\{ {}^t\left(t\frac{d}{dt}\Phi(t)\right)\Phi' - {}^t\Phi(t)\left(t\frac{d}{dt}\Phi'(t)\right)\right\}$$

must vanish identically. For general values of s this should imply that

$$(s+a)\Phi_1\Phi_1' + (-s+a)\Phi_2\Phi_2' - (-s+a)\Phi_1\Phi_2' - (s+a)\Phi_2\Phi_1' = 0$$

or

$$2s\{\Phi_1\Phi_2' - \Phi_2\Phi_1'\} = 0 \ .$$

This appears to be the analogue of the relation on page 170 of Maass's paper on non-analytic automorphic forms. One should be able to use it to prove uniqueness theorems.

If φ lies in $L(\chi, s, \zeta, \pi)$ and φ' lies in $L(\chi, s', \zeta', \pi)$ one should also be able to show that

$$\{\pi_{s,\zeta}(D) - \overline{\pi_{s',\zeta'}(D)}\} \int_{B_x} \{\lambda(D)\varphi\bar{\varphi}' - \varphi\overline{\lambda(D)\varphi'}\}\, dg$$

is equal to

$$(B) \qquad \text{const} \times t^{-2a_0} \left\{ {}^t(t\frac{d}{dt}\Phi(t))(\overline{\Phi}'(t)) - {}^t\Phi(t)\left(\overline{t\frac{d}{dt}\Phi'(t)}\right)\right\}_{t=x} \times {}^t\Psi\bar{\Psi}' \ .$$

Let w be an element of the normalizer of A which takes P to its opposite. Let $\theta(g,s)$ be a function in $L(s,\zeta)$ of the form

$$\theta(g,s) =< a, -s\alpha - \rho > {}^t\Theta\pi(k^{-1})\Xi$$

where Θ lies in $L(\zeta,\pi)$. That is

$$\prod_{i=1}^{\ell}(\pi(Z) - \overline{\zeta_i(Z)})\theta = 0, \quad Z \in \mathfrak{Z}_1 \ .$$

The integral

$$\int_{U_K} \theta(guw,s)\bar{\chi}(u)\, du = \varphi(g,s)$$

is the integral of Jacquet. Thus it should converge for $Re(s) > 0$. Introduce $\Phi(t,s)$ as above; then

$$\Phi(t,s) = t^{s+a_0}(M_1(s)\Theta + O(t)) + t^{-s+a_0}(M_2(s)\Theta + O(t)) \ .$$

For each s, $M_1(s)$ and $M_2(s)$ are linear transformations of $L(s,\pi)$ to itself. If $< a, \alpha >= t^{-1}$ and $\theta(g,s) =< a, -s\alpha - \rho > \bar{\pi}(k)\Theta$ then

$$\begin{aligned}\Phi(ts) &= \int_{U_K} \theta(auw,s)\bar{\chi}(u)\, du \\ &= \int_{U_K} \theta(aua^{-1}w(w^{-1}aw),s)\bar{\chi}(u)\, du \ .\end{aligned}$$

Since $w^{-1}aw = a^{-1}$ this is equal to

$$\langle a, s\alpha - \rho\rangle \int_{U_K} \theta(uw,s)\bar{\chi}(a^{-1}ua)\, du \ .$$

Now $\langle a, s\alpha - \rho\rangle = t^{-s+a_0}$ and, as $t \to 0$, $a^{-1}ua \to 1$. It follows that if $Re(s) > 0$

$$M_2(s)\Theta = \int_{U_K} \theta(uw, s)\, du\ .$$

If π is the trivial representation of K, Harish-Chandra has shown in his paper on spherical functions that $M_2(s)$ can be analytically continued in the whole plane as a meromorphic function. Although he has not published all the details, he can I believe handle the general case. Anyhow assume for the sake of the argument that $M_2(s)$ can be analytically continued. By examining the behaviour of $M_2(s)$ as $s \to \infty$ one can probably show that $\det M_2(s) \not\equiv 0$ so that $M_2(s)^{-1}$ is also meromorphic.

Assume also that the method of Frobenius allows one to show that

$$\Phi(t, s) = t^{s+a_0} N_1(t, s) M_1(s)\Theta + t^{-s+a_0} N_2(t, s) M_2(s)\Theta$$

where

$$N_i(t, s) = I + O(t)$$

uniformly for s in a compact set. (Of course one will probably have to avoid certain exceptional s.) Let

$$t\frac{d}{dt} N_i(t, s) = D_i(t, s)\ .$$

Since $\pi_{s,\zeta}(D) = cs^2 + d$ where d is real and c is positive one can, if s is not real or pure imaginary, take $\varphi(g) = \varphi(g, s)$ and $\varphi'(g) = \varphi'(g, s)$ in formula (B) to see that

$$\begin{aligned}
&-1/(s^2 - \bar{s}^2) \text{ times}\\
&{}^t[\{(s+a_0)t^s N_1(t,s)M_1(s) + t^s P_1(t,s)M_1(s) + (-s+a_0)t^{-s}N_2(t,s)M_2(s) + t^{-s}P_2(t,s)M_2(s)\}\Theta]\\
&k\times\overline{[\{t^s N_1(t,s)M_1(s) + t^{-s}N_2(t,s)M_2(s)\}\Theta']}\\
&-{}^t[\{t^s N_1(t,s)M_1(s) + t^{-s}N_2(t,s)M_2(s)\}\Theta]\\
&\times\overline{[\{(s+a_0)t^s N_1(t,s)M_1(s) + t^s P_1(t,s)M_1(s) + (-s+a_0)t^{-s}N_2(t,s)M_2(s) + t^{-s}P_2(t,s)M_2(s)\}\Theta']}
\end{aligned}$$

is a positive semi-definite hermitian form on Θ and Θ'. This expression equals, if $s = \sigma + i\tau$,

$$\frac{1}{2\sigma}\left[t^{-2\sigma}(N_2(t,s)M_2(s)\Theta, N_2(t,s)M_2(s)\Theta') - t^{2\sigma}(N_1(t,s)M_1(s)\Theta, N_1(t,s)M_1(s)\Theta')\right]$$

minus

$$\frac{1}{2i\tau}\left[t^{2i\tau}(N_1(t,s)M_1(s)\Theta, N_2(t,s)M_2(s)\Theta') - t^{-2i\tau}(N_2(t,s)M_s\Theta, N_1(t,s)M_1(s)\Theta')\right]$$

minus four other terms involving $P_i(t, s)$ of which one would be

$$t^{2\sigma}\left[{}^t\Theta\, {}^tM_1(s)\left\{\frac{{}^tP_1(t,s)\bar{N}_1(t,s) - {}^tN_1(t,s)\bar{P}_1(t,s)}{s^2 - \bar{s}^2}\right\}\bar{M}_1(s)\bar{\Theta}'\right]\ .$$

If the last four terms were not present this expression would be just like the one used to effect the analytic continuation of Eisenstein series. Since the last four terms are $O(t)$ it is not inconceivable that the same techniques could be used to handle Jacquet's integral.

It is going to be more difficult to handle the continuation when K is non-archimedean and, at the moment, I have no suggestions. It seems to me however that when G is quasi-split, split over an unramified extension, D is trivial, and the character ζ is generic it should be possible to obtain a simple closed expression for what Jacquet calls the Whittaker function. The only case I have thought about is the adjoint group of a Chevalley group over $\mathbb{Q}_p$.

Let w be an element of the Weyl group which takes all positive roots to negative roots. Suppose ζ is such that the μ_α on page 44 of Jacquet are all 1. Rather than the Whittaker function consider

$$\theta(g, \lambda) = \int_{U_K} L(guw, \lambda + \rho)\overline{\zeta(u)}\, du \ .$$

Certainly $\theta(g, \lambda)$ satisfies

(i) $\theta(gu, \lambda) = \zeta(u)\theta(g, \lambda) \quad u \in U_K$

(ii) $\theta(mg, \lambda) = \theta(g, \lambda) \quad m \in M$

Any function which satisfies (i) and (ii) is determined by its restriction to A_K. Moreover if $u \in U_K \cap M$ then

$$\theta(a, \lambda) = \theta(ua, \lambda) = \zeta(a^{-1}ua)\theta(a, \lambda) \ .$$

Consequently $\theta(a, \lambda) = 0$ unless $\langle a, \alpha\rangle \geq 1$ for all positive roots so that a lies in the 'positive Weyl chamber'. By the way using this one should be able to show by an inductive argument that if a function satisfies (i) and (ii), is an eigenfunction of the Hecke algebra, and vanishes at 1 then it vanishes identically. This would be a simple uniqueness theorem.

Set

$$\omega(g, \lambda) = \prod_{\alpha > 0} p^{\lambda(H_\alpha)/2} \left\{ \frac{1 - \dfrac{1}{p^{\lambda(H_\alpha)}}}{1 - \dfrac{1}{p^{\lambda(H_\alpha)+1}}} \right\} \theta(g, \lambda) \ .$$

The functional equation of Jacquet is just

$$\omega(\sigma\lambda, g) = \operatorname{sgn} \sigma\ \omega(\lambda, g)$$

if σ is in the Weyl group. (He does not work with the adjoint group but that is no matter.) The function $\omega(g, \lambda)$ appears to be an entire function of λ. Thus it can be expanded in a Fourier series. The resulting formula will be more poignant if one makes a simple observation first. Consider the following objects.

$$
\begin{array}{cc}
\{\hat{\alpha}_1 = H_{\alpha_i} | \alpha_i \text{ a simple root}\} & \{\alpha_i = H_{\hat{\alpha}_i} | \alpha_i \text{ a simple root}\} \\
\bigcap| & \bigcap| \\
\{\hat{\alpha} = H_{\alpha} | \alpha \text{ a root}\} & \{\alpha = H_{\hat{\alpha}} | \alpha \text{ a root}\} \\
\bigcap| & \bigcap| \\
\mathrm{Hom}\,(L, \mathbb{Z}) = \hat{L}' & L' = \text{ lattice spanned by roots} \\
\bigcap| & \bigcap| \\
\mathrm{Hom}\,(L', \mathbb{Z}) = \hat{L} & L \text{ integral linear functions} \\
\bigcap| & \bigcap| \\
\text{Cartan subalgebra of } g_{\mathbb{Q}} = h_{\mathbb{Q}} & \hat{h}_{\mathbb{Q}} = \text{ dual of } h_{\mathbb{Q}} \\
\bigcap| & \bigcap| \\
h_{\mathbb{R}} & \hat{h}_{\mathbb{R}} \\
\bigcap| & \bigcap| \\
h_{\mathbb{C}} & \hat{h}_{\mathbb{C}}
\end{array}
$$

There is another semi-simple group $\widehat{G}$ over $\mathbb{Q}$ with a Cartan subalgebra which may be identified with $\widehat{h}_{\mathbb{Q}}$ so that the roots correspond to $\{\widehat{\alpha}\}$. Then there is a duality between the two columns. Moreover there is an isomorphism $a \to \widehat{\lambda}(a)$ of A_K/A_O with $\widehat{L}$ which is such that $\langle a, \lambda\rangle = p^{\langle \lambda, \widehat{\lambda}(a)\rangle}$ if $\langle \lambda, \widehat{\lambda}\rangle$ is the pairing of $h_{\mathbb{Q}}$ and $\widehat{h}_{\mathbb{Q}}$. Then

$$\omega(a, \lambda) = \left\{ \prod_{\alpha > 0} \frac{p^{\langle \lambda, \widehat{\alpha}\rangle/2} - p^{-\langle \lambda, \widehat{\alpha}\rangle/2}}{1 - \dfrac{1}{p^{\langle \lambda, \widehat{\alpha}\rangle + 1}}} \right\} \theta(a, \lambda)$$

and

$$\omega(a, \lambda) = \sum\nolimits_{\widehat{\lambda} \in \widehat{L}} \gamma'(a, \widehat{\lambda}) p^{\langle \lambda, \widehat{\lambda}\rangle}$$

Because of the skew symmetry the right side may be written as

$$\sum\nolimits_{\widehat{\lambda} \in \widehat{L}^+} \gamma(a, \widehat{\lambda}) \sum_{\widehat{\sigma} \in \text{ Weyl group}} \operatorname{sgn} \widehat{\sigma} p^{\langle \lambda, \widehat{\sigma}(\widehat{\lambda} + \widehat{\rho})\rangle}$$

if $\widehat{L}^+$ is the intersection of $\widehat{L}$ with the positive Weyl chamber. It seems to be the case that if $n \in U_K$ and $m_1 a_1 n_1$ is the Iwasawa decomposition of nw then $\langle \widehat{\lambda}(a_1), \alpha\rangle \geq 0$ if $\alpha > 0$ and $\widehat{\lambda}(a_1) = 0$ if and only if $n \in M$. It follows that $\gamma(a, \widehat{\lambda}) = 0$ unless $\langle \alpha, \widehat{\lambda}(a) - \widehat{\lambda}\rangle \geq 0$ for all $\alpha > 0$ and that

$$\gamma(a, \widehat{\lambda}(a)) = p^{-\langle \rho, \widehat{\lambda}(a)\rangle} \ .$$

I would not be surprised if $\gamma(a, \widehat{\lambda})$ were 0 for $\widehat{\lambda} \neq \widehat{\lambda}(a)$.

For your amusement I would like to mention a possible application of the ideas of Jacquet's thesis. Unfortunately it will not work until his results are improved slightly. Suppose G^0 is the adjoint group of a Chevalley group over $\mathbb{Q}$ and suppose ϕ is a cusp form on $G^0_{\mathbb{A}}/G^0_{\mathbb{Q}}$ which is invariant under M and is an eigenfunction of the

Hecke algebra H_p at every prime including infinity. Let χ_p be the homomorphism of the Hecke algebra into $\mathbb{C}$ associated to p. If p is finite then H_p is isomorphic to the representation ring of the simply connected group $\widehat{G}^0$. Thus there is a semi-simple conjugacy class $\{g_p\}$ in $\widehat{G}^0_{\mathbb{C}}$ such that if f in H_p corresponds to the representation π then $\chi_p(f) = \mathrm{trace}\pi(g_p)$. To χ_∞ one can associate a semi-simple conjugacy class in the Lie algebra $\widehat{\mathfrak{g}}^0_{\mathbb{C}}$. If π is a representation of G^0 choose X_∞ in diagonal form and set

$$\Gamma(s,\pi,\phi) = \prod_{\widehat{\lambda}\in\widehat{L}} \left\{ \pi^{-(s-\widehat{\lambda}(X_\infty)/2}\Gamma\left(\frac{s-\widehat{\lambda}(X_\infty)}{2}\right)\right\}^{m_{\widehat{\lambda}}}$$

where $m_{\widehat{\lambda}}$ is the multiplicity with which $\widehat{\lambda}$ occurs in π. If

$$\xi(s,\pi,\phi) = \Gamma(s,\pi,\phi) \prod_{p \text{ finite}} \frac{1}{\det\left(I - \frac{\pi(g_p)}{p^s}\right)}$$

then $\xi(s,\pi,\phi)$ is analytic in a half-plane. Beyond that nothing is known for general π. Let me show you how the ideas of Jacquet might be applied to obtain a functional equation for particular choices of ϕ and π.

Suppose G is also the adjoint group of a Chevalley group over $\mathbb{Q}$, $P \supseteq B$ a parabolic subgroup of rank one, and $P = ZN$ the decomposition of P as the product of a reductive Z and the unipotent radical N. Let C be the centre of Z and suppose that $G^0 \cong Z/C \cong P/NC$. Because of the map $P_{\mathbb{A}} \to G^0_{\mathbb{A}}$ one can regard ϕ as a function on $P_{\mathbb{A}}$. If $p \in P$ let $\chi(p)$ be the determinant of $\mathrm{Ad}(p)$ acting on $\mathfrak{n}$, the Lie algebra of N, and set

$$F(g,s,\phi) = |\chi(b)|^{-s-1/2}\,\phi(b)$$

if $g = mb$, $m \in M$, $b \in P_{\mathbb{A}}$. The Eisenstein series is

$$E(g,s,\phi) = \sum_{G_{\mathbb{Q}}/P_{\mathbb{Q}}} F(g\gamma,s,\phi)\ .$$

Let $'P \supseteq B$ be conjugate to the parabolic group opposed to P. Let ω_0 in the normalizer of A in M be such that $\omega_0{}'P\omega_0^{-1}$ is opposed to P. Let $Z' = \omega_0^{-1}Z\omega_0$, $C' = \omega_0^{-1}C\omega_0$, and set $'G^0 = Z'/C'$. Let ϕ' be he function on $'G^0$ defined by

$$\phi'(m') = \phi(\omega_0 m' \omega_0^{-1})\ .$$

The functional equation connects $E(g,s,\phi)$ and $E(g,-s,\phi')$. To describe it complete the two columns on page 18 by

$$\begin{array}{ccl} G & \widehat{G} & \text{- simply connected} \\ \cup & \cup & \\ P & \widehat{P} & \\ \downarrow & \downarrow & \\ G^0 & \widehat{G}^0 & \text{- simply connected} \end{array}$$

Let $\widehat{\mathfrak{n}}$ be spanned by $\{X_{\widehat{\alpha}} \mid \widehat{\alpha} \in \theta\}$ and set

$$H_0 = \sum_{\widehat{\alpha}\in\theta} H_{\widehat{\alpha}} \; .$$

Let $a_1, \ldots, a_r$ be the eigenvalues of H_0 acting on $\widehat{\mathfrak{n}}$ and let $\widehat{\mathfrak{n}}_1, \ldots, \widehat{\mathfrak{n}}_r$ be the corresponding subspaces. Let π_i be the representation of $\widehat{G}^0$ on $\widehat{\mathfrak{n}}_i$ and $\widetilde{\pi}_i$ its contragredient. Then

$$E(g,s,\phi) = \left\{ \prod_{i=1}^{r} \frac{\xi(a_i s, \widetilde{\pi}_i, \phi)}{\xi(a_i s+1, \widetilde{\pi}_i, \phi)} \right\} E(g,-s,\phi') \; .$$

Using this one can show, at least for simple groups, that $\xi(s,\pi_i,\phi)$ and $\xi(s,\widetilde{\pi}_i,\phi)$ are meromorphic in the whole plane.

Suppose ξ is a character of $U_{\mathbb{A}}/U_{\mathbb{Q}}$ so that the μ_α on page 93 of Jacquet are all 1. Set

$$\eta(g,s,\phi) = \int_{U_{\mathbb{A}}/U_{\mathbb{Q}}} E(gu,s,\phi)\bar{\xi}(u)\, du \; .$$

What I want to point out is that the results of Jacquet are almost good enough to show that

$$\eta(g,s,\phi) = \left\{ \prod_{i=1}^{r} \frac{1}{\xi(a_i s+1, \widetilde{\pi}_i, \phi)} \right\} \mu(g,s,\phi)$$

$$\eta(g,s,\phi') = \left\{ \prod_{i=1}^{r} \frac{1}{\xi(a_i s+1, \pi_i, \phi)} \right\} \mu'(g,s,\phi')$$

with $\mu(g,s,\phi) = \mu'(g,-s,\phi')$. For many ϕ the function $\mu(g,s,\phi)$ will not vanish identically. Then one deduces that

$$\prod_{i=1}^{r} \xi(a_i s, \widetilde{\pi}_i, \phi) = \prod_{i=1}^{r} \xi(1-a_i s, \pi_i, \phi)$$

and then, at least for simple groups,

$$\xi(s,\widetilde{\pi}_i,\phi) = \xi(1-s,\pi_i,\phi) \; .$$

I leave the calculations to Jacquet if he is interested. The biggest obstacle will be a uniqueness theorem at the infinite prime.

I apologize for the technicality of this letter. I felt that the best way I had of indicating the interest of Jacquet's ideas would be to explain, as well as I could, their possible implications for the theory of group representations and automorphic forms. Unfortunately, although I had been thinking about the implications since I first noticed Jacquet's papers I had written nothing down. Because I had to hurry all my suggestions had to be tentative. I hope there is something in them of value which has not occurred to you or Jacquet.

Yours truly,

R. Langlands

Let me add some remarks not related to Jacquet's thesis. Since you were rather skeptical about the interest of the functions $\xi(s,\pi,\phi)$ when I spoke to you in Princeton I would like to comment on their relation to a generalized

Ramanujan conjecture. For the ordinary Ramanujan conjecture one has to consider functions which are not M invariant. I have not yet tried to understand the complications this entails. Let $\widehat{U}^0$ be a maximal compact subgroup of $\widehat{G}^0_{\mathbb{C}}$. The generalized Ramanujan conjecture for the function ϕ on p. 20 would say that when p is finite the conjugacy class $\{g_\rho\}$ meets $\widehat{U}^0$ and that the conjugacy class $\{X_\infty\}$ meets the Lie algebra of $\widehat{U}^0$. It implies that $\Gamma(s,\pi,\phi)$ is analytic for $Re\, x > 0$ and that the Euler product on p. 20 converges for $Re\, s > 1$ so that $\xi(s,\pi,\phi)$ is analytic for $Re\, s > 1$.

Conversely suppose that, for all π, $\xi(s,\pi,\phi)$ is analytic for $Re\, s > 1$. If p is finite then H_1 is isomorphic to the representation ring of $\widehat{G}$ over $\mathbb{C}$. The involution $f \to \tilde{f}$ with $\tilde{f}(g) = \bar{f}(g^{-1})$ corresponds to the involution $\Sigma a_\rho \rho \to \Sigma \overline{a}_\rho \tilde{\rho}$. Since $\chi_\rho(\tilde{f}) = \overline{\chi}_\rho(f)$ one has trace $\tilde{\rho}(g_\rho) = \overline{\text{trace } \rho(\mathrm{g}_\rho)}$. In the same way the eigenvalues of $\tilde{\rho}(X)$ are the complex conjugates of those of $\rho(X)$. Take a representation ρ and let $\pi = \rho \otimes \tilde{\rho}$. Since $\xi(s,\pi,\phi)$ is analytic for $Re\, s > 1$ and the Γ-function has no zeros so is

$$L(s,\pi,\phi) = \Pi_p \text{ finite} \frac{1}{\det\left(L - \dfrac{\pi(g_p)}{p^s}\right)}.$$

Observe that the coefficients of the Dirichlet series log $L(s,\pi,\phi)$ and hence those of $L(s,\pi,\phi)$ are positive because

$$\log L(s,\pi,\phi) = \Sigma_1 \Sigma_{n=1}^{\alpha} \frac{\text{trace } \pi^n(g_\rho)}{\rho^{ns}}$$

and

$$\text{trace } \pi^n(g_\rho) = \text{ trace } \rho^n(g_0) \cdot \text{ trace } \tilde{\rho}^n(g_\rho) = |\text{ trace } \rho^n(g_0)|^2$$

By Landau's theorem the series converges absolutely for $Re\, s > 1$. In particular

$$\det\left(1 - \frac{\pi(g_\rho)}{\rho^s}\right)$$

does not vanish for $Re\, s > 1$ so that the eigenvalues of $\pi(g_\rho)$ are all less than or equal to ρ in absolute value. Choose g_ρ in the Cartan subgroup $\widehat{A}^0_{\mathbb{C}}$ and let $\widehat{\lambda}$ be a weight. Given an integer m' choose ρ so that $m\widehat{\lambda}$ occurs in ρ. Then $(\xi_{\widehat{\lambda}}(g_\rho))^m$ is an eigenvalue of $\rho(g_\rho)$ so $\overline{(\xi_{\widehat{\lambda}}(g_\rho))^n}$ is one eigenvalue of $\tilde{\rho}(g_\rho)$ and $|\xi_{\widehat{\lambda}}(g_\rho)|^{2m}$ is an eigenvalue of $\pi(g_\rho)$. Consequently

$$p^{-1/2m} = |\xi_{\widehat{\lambda}}(g_\rho)| \leq p^{1/2m}$$

for all $m > 0$. This takes care of the finite primes.

Choose X_∞ in the Cartan subalgebra. Since $L(s,\pi,\phi)$ cannot vanish for $Re\, s > 1, \Gamma(s,\pi,\phi)$ will be analytic in this region. From the expression for $\Gamma(s,\pi,\phi)$ one concludes that all the eigenvalues of $\pi(X_\infty)$ have real part at most 1. Again given $\widehat{\lambda}$ choose ρ so that in $\widehat{\lambda}$ is a weight of ρ. Then $2mRe\,(\widehat{\lambda}(X_\infty))$ is an eigenvalue of π. Thus

$$-\frac{1}{2m} \leq Re\, \widehat{\lambda}(X_\infty) \leq \frac{1}{2m}$$

for all $m > 0$. This takes care of the infinite prime.

The series also seem to be related to Sato's conjecture. I too have been thinking a little about Weil's paper. I plan to spend the next month getting my thoughts organized. It will take me a while to digest the second part of your letter. Do you still plan to come to Princeton for 1968-1969? Unfortunately Hunt put me in such a position that I had no choice but to resign from Princeton. I will be going to Yale when I return from Ankara so I shall not see much of you if you come. As you probably know I received an invitation from Lions to spend a month in Paris either next year or the year after. I would like to ask your advice. I gave some lectures (I will send you the notes) at Yale last spring on the relation of the functions $\xi(s,\pi,\phi)$ to Eisenstein series. I would like, for my own information, to work things out for general automorphic forms on general reductive groups. After that I want to try to formulate exactly the definition of $\xi(s,\pi,\phi)$ in the general case. Then I have to try to prove something. Since I don't have the foggiest idea how to proceed at the moment it might be a while (never?) before I get anything interesting. Since I would be expected to give some lectures in Paris I should have something to say. Do you think that I could write to Lions and say that I should like very much to come but do not know when I could be prepared; that I would tell him when I was prepared and then if he still cared to he could renew the invitation. It would also be best for me to come when you are there.

Yours truly,

R. Langlands