Euler Products

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EULER PRODUCTS [†]

James K. Whittemore Lectures in Mathematics given at Yale University

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1. INTRODUCTION

The first undisguised automorphic forms met by most of us are the modular forms. A modular form of weight k is an analytic function on the upper half-plane which satisfies

$$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z)$$

for all integral matrices

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

of determinant 1. Actually there is a supplementary condition, of no importance to us, on its rate of growth as $\mathrm{Im}(z) \to \infty$. Nowadays the Siegel modular forms are met soon afterwards. A Siegel modular form of weight k is a complex analytic function on the space of complex $n \times n$ symmetric matrices with positive definite real part which satisfies

$$f\big((AZ+B)(CZ+D)^{-1}\big) = \det(CZ+D)^k f(Z)$$

for all integral $2n \times 2n$ symplectic matrices

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

For some purposes it is best to consider not f but an associated function ϕ_f on the group G of real $2n \times 2n$ symplectic matrices defined by

$$\phi_f(g) = \det(Ci+D)^{-k} f\big((Ai+B)(Ci+D)^{-1}\big)$$

if

$$g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

If Γ is the group of integral matrices in G then ϕ_f is a function on $\Gamma\backslash G$. The functions ϕ_f associated to those f which have a finite norm in the Petersson metric can be characterized in terms of the representations of G. Associating to each h in G the operator $\lambda(h)$ on $L^2(\Gamma\backslash G)$ defined by

$$\big(\lambda(h)\phi\big)(g) = \phi(gh),$$

we obtain a representation of G on $L^2(\Gamma\backslash G)$. There is a representation π_k of G on a Hilbert space H_k and a distinguished subspace H_k^0 of H_k such that ϕ is a ϕ_f if and only if there is a G-invariant map of H_k to $L^2(\Gamma\backslash G)$ which takes H_k^0 to the space generated by ϕ.

This is the first hint that it might not be entirely unprofitable to study automorphic forms in the context of the theory of group representations. The Hecke operators, which play a major role in the study of modular forms, provide a second. To see this we have to introduce the adèle group of $2n \times 2n$ symplectic matrices. It will be convenient to change the notation a little. If R is a commutative ring let G_R be the group of $2n \times 2n$ symplectic matrices with entries from R. Thus the groups Γ and G become $G_{\mathbb{Z}}$ and $G_{\mathbb{R}}$. If p is a prime, finite or infinite, let $\mathbb{Q}_p$ be the corresponding completion of $\mathbb{Q}$. The adèle ring $\mathbb{A}$ is the set of elements $\{a_p\}$ in $\prod_p \mathbb{Q}_p$ such that a_p is integral for all but a finite number of p. The diagonal map defines an imbedding of $\mathbb{Q}$ in $\mathbb{A}$. There is a standard topology on $G_{\mathbb{A}}$ which turns it into a locally compact group with $G_{\mathbb{Q}}$ as a discrete subgroup. Let

$$\mathbb{A}^0 = \mathbb{R} \times \prod_{p \text{ finite}} \mathbb{Z}_p\,.$$

It is known that $G_{\mathbb{A}} = G_{\mathbb{Q}} G_{\mathbb{A}^0}$. Any function ϕ on $G_{\mathbb{Z}} \backslash G_{\mathbb{R}}$ can be regarded as a function on $G_{\mathbb{Q}} \backslash G_{\mathbb{A}}$ if one sets $\phi(g) = \phi(g_2')$ when $g = g_1 g_2,\ g_1 \in G_{\mathbb{Q}},\ g_2 \in G_{\mathbb{A}^0}$, and g_2' is the projection of g_2 on $G_{\mathbb{R}}$. The functions so obtained are characterized by their right invariance under

$$U = 1 \times \prod_{p \text{ finite}} G_{\mathbb{Z}_p} .$$

If f is a function, with compact support, on

$$G^0 = \left\{1 \times \prod_{p \text{ finite}} G_{\mathbb{Q}_p}\right\} \cap G_{\mathbb{A}} ,$$

which is invariant on the left and right under U, and if $\lambda(f)\phi$ is defined by

$$\lambda(f)\phi(g) = \int_{G^0} \phi(gh) f(h)\, dh ,$$

then $\lambda(f)\phi$ is invariant on the right under U if ϕ is. Thus the operators $\lambda(f)$, the Hecke operators, act on the functions on $G_{\mathbb{Z}} \backslash G_{\mathbb{R}}$. If ϕ belongs to a subspace H of $L^2(G_{\mathbb{Q}} \backslash G_{\mathbb{A}})$ which is invariant and irreducible under the action of $G_{\mathbb{A}}$ then ϕ is an eigenfunction of all the Hecke operators and the corresponding eigenvalues are determined by the equivalence class of the representation of $G_{\mathbb{A}}$ on H.

The theory of modular forms and the operators $\lambda(f)$ is far from complete. Indeed very little attempt has been made, so far as I can see, to understand what the goals of the theory should be. Once it is put in the above form it is clear that the concepts at least admit of extension to any reductive algebraic group defined over a number field. It may be possible to give some coherency to the subject by introducing the simple principle, implicit in the work of Harish-Chandra, that what can be done for one reductive group should be done for all.

The simplest reductive group over $\mathbb{Q}$ is $GL(1)$. Since $G_{\mathbb{A}}$ is abelian, the space $G_{\mathbb{Q}} \backslash G_{\mathbb{A}}$ is a locally compact abelian group C, the group of idèle classes of $\mathbb{Q}$. According to the Plancherel theorem for abelian groups the characters of C can be used to decompose $L^2(G_{\mathbb{Q}} \backslash G_{\mathbb{A}})$ under the action of $G_{\mathbb{A}}$ and the characters of C must be regarded as the basic automorphic forms.

Since $\mathbb{Q}_p^\times$, the multiplicative group of $\mathbb{Q}_p$, is contained in $G_{\mathbb{A}}$, each character χ of C defines a character χ_p of $\mathbb{Q}_p^\times$. If p does not divide f_χ, the conductor of χ, χ_p is trivial on the units of $\mathbb{Q}_p^\times$. It is customary to associate to χ the L-series

$$L(s, \chi) = \prod_{p \nmid f_\chi} \frac{1}{1 - \dfrac{\chi_p(p)}{p^s}}$$

and the function

$$\xi(s, \chi) = \left(\frac{\pi}{f_\chi}\right)^{-(s+\alpha+\beta)/2} \Gamma\left(\frac{s + \alpha + \beta}{2}\right) L(s, \chi)$$

if

$$\chi_\infty(x) = (\operatorname{sgn} x)^\alpha |x|^\beta .$$

The number α is 0 or 1. The function $\xi(s, \chi)$ is known to be meromorphic in the entire complex plane and to satisfy

$$\xi(s,\ \chi) = \epsilon(\chi)\xi(1 - s,\ \chi^{-1})$$

where $\epsilon(\chi)$ is a constant of absolute value 1.

For a general group $L^2(G_{\mathbb{Q}} \backslash G_{\mathbb{A}})$ decomposes not into a direct integral of one-dimensional spaces but into a direct integral of Hilbert spaces on each of which $G_{\mathbb{A}}$ acts irreducibly. Can we associate to each of these Hilbert spaces an Euler product with the same analytic properties as the functions $L(s,\ \chi)$? In these lectures I would like to present a little evidence, far from conclusive, that the answer is “yes”. Let me observe that each power of a character is also a character, thus to χ is associated the whole collection $\{L(s, \chi^n) \mid n \in \mathbb{Z}\}$, and that $\mathbb{Z}$ parametrizes not only the powers of χ but also the rational representations of $GL(1)$.

Before beginning the substantial part of these lectures let me make, without committing myself, a further observation. The Euler products mentioned above are defined by means of the Hecke operators. Thus they are defined in an entirely different manner than those of Artin or Hasse-Weil. An assertion that an Euler product of the latter type is equal to one of those associated to an automorphic form is tantamount to a reciprocity law (for one equation in one variable in the case of the Artin L-series and for several equations in several variables in the case of the Hasse-Weil L-series).

2. SOME EULER PRODUCTS

Suppose $\mathfrak{g}$ is a split semisimple Lie algebra over $\mathbb{Q}$ and G its adjoint group. Fix a split Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ and choose a Chevalley basis for $\mathfrak{g}$ with respect to $\mathfrak{h}$. Let M be the lattice generated by the Chevalley basis over $\mathbb{Z}$. If p is a finite prime let $G_{\mathbb{Z}_p}$ be the stabilizer of $M \otimes \mathbb{Z}_p$ in $G_{\mathbb{Q}_p}$. If p is the infinite prime let

$$G_{\mathbb{Z}_p} \subseteq G_{\mathbb{Q}_p} \equiv G_{\mathbb{R}}$$

be the maximal compact subgroup of $G_{\mathbb{R}}$ corresponding to the involution associated to the Chevalley basis. Let

$$U = \prod_p G_{\mathbb{Z}_p} \subseteq G_{\mathbb{A}}\,,$$

be the adèle group of G. Fix a Borel subgroup B containing T, the Cartan subgroup with Lie algebra $\mathfrak{h}$. It is known that

$$G_{\mathbb{Q}_p} = B_{\mathbb{Q}_p} G_{\mathbb{Z}_p}$$

for each p. For this and various other facts about reduction theory over local fields see the article by F. Bruhat in the collection *Algebraic Groups and Discontinuous Subgroups*. As a consequence $G_{\mathbb{A}} = B_{\mathbb{A}} U$. Moreover $B_{\mathbb{A}} = B_{\mathbb{Q}} B_{\mathbb{R}} (B_{\mathbb{A}} \cap U)$; hence $G_{\mathbb{A}} = B_{\mathbb{Q}} G_{\mathbb{R}} U$ and $G_{\mathbb{Q}} = B_{\mathbb{Q}} G_{\mathbb{Z}}$ if $G_{\mathbb{Z}}$ is the stabilizer of M. In particular any function on $G_{\mathbb{Q}} \backslash G_{\mathbb{A}} / U$ is determined by its restriction to $G_{\mathbb{R}}$.

Let L be the space of all square integrable functions on $G_{\mathbb{Q}} \backslash G_{\mathbb{A}}$ which are invariant under right translations by elements of U. Let P be a parabolic group containing B and let N be its unipotent radical. If ϕ lies in L then

$$\int_{N_{\mathbb{Q}} \backslash N_{\mathbb{A}}} \phi(ng)\, dn$$

vanishes for almost all g in $G_{\mathbb{A}}$ if and only if

$$\int_{N_{\mathbb{Z}} \backslash N_{\mathbb{R}}} \phi(ng)\, dn$$

vanishes for almost all g in $G_{\mathbb{R}}$. If these integrals vanish for almost all g for all choices of P except G itself we say that ϕ is a cusp form. The set of cusp forms is a closed subspace L_0 of L.

If p is a prime, finite or infinite, let H_p be the algebra of all compactly supported regular Borel measures on $G_{\mathbb{Q}_p}$ invariant under left and right translations by elements of $G_{\mathbb{Z}_p}$. Multiplication is given by convolution. If μ lies in H_p, define the operator $\lambda(\mu)$ on L_0 by

$$\lambda(\mu)\phi(g) = \int_{G_{\mathbb{Q}_p}} \phi(gh)\, d\mu(h)\,.$$

If μ is the measure associated with an L^1 function f we shall sometimes write $\lambda(f)$ instead of $\lambda(\mu)$. Moreover f will be regarded as an element of H_p. If p is finite all the measures in H_p are absolutely continuous with respect to Haar measure. There is an orthonormal basis $\phi_1, \phi_2, \ldots$ of L_0 such that each ϕ_i is, for all p, an eigenfunction of $\lambda(\mu)$ for all μ in H_p.

Fix one element ϕ of this basis. If μ belongs to H_p let $\lambda(\mu)\phi = \chi_p(\mu)\phi$. The map $\mu \to \chi_p(\mu)$ is a homomorphism of H_p into the complex numbers. Let me remind you of the standard method of obtaining all such homomorphisms. Let V be the unipotent radical of B. Now $V_{\mathbb{Q}_p} \backslash B_{\mathbb{Q}_p}$ is isomorphic to $T_{\mathbb{Q}_p}$. Thus any homomorphism w of $T_{\mathbb{Q}_p}/T_{\mathbb{Z}_p}$ into the complex numbers determines a homomorphism of $B_{\mathbb{Q}_p}$ into the complex numbers which we again denote by w. If b belongs to B let $\xi(b)$ be the determinant of the restriction of $\mathrm{Ad}(b)$ to $\mathfrak{v}$, the Lie algebra of V. If g lies in $G_{\mathbb{Q}_p}$ then g can be written as a product bk with b in $B_{\mathbb{Q}_p}$ and k in $G_{\mathbb{Z}_p}$. Set

$$\psi_w(g) = w(b)|\xi(b)|^{\frac{1}{2}}\,.$$

The function ψ_w is well defined and any other function ψ on $G_{\mathbb{Q}_p}$ satisfying

$$\psi(bgk) = w(b)|\xi(b)|^{\frac{1}{2}}\psi(g) \tag{1}$$

for all $b,\ g$, and k is a scalar multiple of ψ_w. If μ lies in H_p define $\lambda(\mu)\psi_w$ by

$$\big(\lambda(\mu)\psi_w\big)(g) = \int_{G_{\mathbb{Q}_p}} \psi_w(gh)\,d\mu(h)\,.$$

The function $\lambda(\mu)\psi_w$ satisfies (1) so there is a scalar $\chi_w(\mu)$ such that

$$\lambda(\mu)\psi_w = \chi_w(\mu)\psi_w\,.$$

The map $\mu \to \chi_w(\mu)$ is a homomorphism of H_p into $\mathbb{C}$. All homomorphisms of H_p into the complex numbers which are continuous in the weak topology are obtained in this way. The homomorphism χ_w equals $\chi_{w'}$ if and only if there is a σ in the Weyl group such that $w(t) = w'(t^\sigma)$ for all t in $T_{\mathbb{Q}_p}$.

Suppose p is finite. If L is the lattice generated by the roots of $\mathfrak{h}$ there is a homomorphism λ from $\overline{T} = T_{\mathbb{Q}_p}/T_{\mathbb{Z}_p}$, or from $T_{\mathbb{Q}_p}$, to ${}^cL = \mathrm{Hom}(L,\,\mathbb{Z})$ such that $|\xi_\alpha(t)| = p^{\lambda(t)(\alpha)}$ if α is a root. Here ξ_α is the character of T associated to α. If α is a root let H_α be, in the language of Chevalley, the *copoid* attached to α. Let $\alpha_1,\ldots,\alpha_n$ be the simple roots. The matrix

$$(A_{ij}) = \left(\frac{(\alpha_i,\,\alpha_j)}{(\alpha_i,\,\alpha_i)}\right)$$

is the Cartan matrix of $\mathfrak{g}$. The matrix

$$\left(\frac{(H_{\alpha_i},\,H_{\alpha_j})}{(H_{\alpha_i},\,H_{\alpha_i})}\right)$$

in the transpose of (A_{ij}) and the Cartan matrix of another Lie algebra ${}^c\mathfrak{g}$. The lattice ${}^cL'$ generated by the roots of a split Cartan subalgebra ${}^c\mathfrak{h}$ of ${}^c\mathfrak{g}$ can be identified with the lattice in $\mathfrak{h}_{\mathbb{R}}$ generated by $H_{\alpha_1},\ldots,H_{\alpha_n}$ in such a way that the roots of ${}^c\mathfrak{h}$ correspond to the elements H_α. Moreover ${}^cL$ can be regarded as a lattice in $\mathfrak{h}_{\mathbb{R}}$. It contains ${}^cL'$ and can in fact be regarded as the lattice of weights of ${}^c\mathfrak{h}$ so $\mathfrak{h}_{\mathbb{R}} \supseteq {}^cL \supseteq {}^cL'$. In the same way ${}^c\mathfrak{h}_{\mathbb{R}}$ may be identified with Hom $({}^cL,\,\mathbb{R})$ so ${}^c\mathfrak{h}_{\mathbb{R}} \supseteq L' \supseteq L$ if L' is the lattice of weights of $\mathfrak{h}$. Let ${}^cG$ be the simply connected group with Lie algebra ${}^c\mathfrak{g}$ and let ${}^cT$ be the Cartan subgroup corresponding to ${}^c\mathfrak{h}$. There is an isomorphism $\sigma \to {}^c\sigma$ of the Weyl group of T in G with that of ${}^cT$ in ${}^cG$ such that

$${}^c\sigma\big(\lambda(t)\big) = \lambda(\sigma t), \quad t \in T_{\mathbb{Q}_p}\,.$$

If w is a homomorphism of $\overline{T}$ into the complex numbers there is a unique point g in ${}^cT_{\mathbb{C}}$ such that $w(t) = \xi_\lambda(g)$ for all t. Here $\lambda = \lambda(t)$ and ξ_λ is the rational character of ${}^cT$ associated to λ. Thus associated to each homomorphism of H_p into the complex numbers is an orbit of the Weyl group in ${}^cT_{\mathbb{C}}$ or, as I prefer, a semisimple conjugacy class in ${}^cG_{\mathbb{C}}$.

The automorphic form ϕ determined for each p a homomorphism χ_p of H_p into $\mathbb{C}$. If p is a finite prime let $\{g_p\}$ be the conjugacy class in ${}^cG_{\mathbb{C}}$ corresponding to χ_p. If π is a finite dimensional complex representation of ${}^cG_{\mathbb{C}}$, we can consider the Euler product

$$\prod_p \frac{1}{\det\left(1 - \dfrac{\pi(g_p)}{p^s}\right)} = L(s,\pi,\phi)\,. \tag{2}$$

As we shall see, this Euler product is absolutely convergent for $\mathrm{Re}\,(s)$ sufficiently large. I do not know in general what the analytic properties of the function $L(s,\pi,\phi)$ are. I will show, however, that for all but three of the simple groups there is at least one nontrivial representation for which $L(s,\pi,\phi)$ is meromorphic in the whole complex plane. For some groups there are several such representations.

Let me first introduce a Γ-factor to go with $L(S,\,\pi,\,\phi)$. If p is the infinite prime there is a homomorphism λ of

$$\overline{T} = T_{\mathbb{Q}_p}/T_{\mathbb{Q}_p} \cap G_{\mathbb{Z}_p}$$

into $\mathfrak{h}_{\mathbb{R}} = \mathrm{Hom}\,(L,\,\mathbb{R})$ such that $|\xi_\alpha(t)| = e^{\lambda(t)(\alpha)}$ if α is a root. Since L is a lattice in ${}^c\mathfrak{h}_{\mathbb{R}}$, every homomorphism of $\overline{T}$ into C is of the form

$$w(t) = e^{\lambda(t)(X)}$$

for some X in ${}^c\mathfrak{h}_{\mathbb{C}}$. Thus to every homomorphism of H_p into $\mathbb{C}$ there is associated an orbit of the Weyl group in ${}^c\mathfrak{h}_{\mathbb{C}}$ or a semisimple conjugacy class in ${}^c\mathfrak{g}_{\mathbb{C}}$. If χ_p is the homomorphism associated to the automorphic form ϕ, let $\{X\}$ be the associated conjugacy class and let $\Gamma(s,\pi,\phi)$ be the inverse of

$$\pi^{\mathrm{trace}\left(\frac{s-\pi(X)}{2}\right)}\det\left(\frac{s-\pi(X)}{2}\right)e^{\gamma\;\mathrm{trace}\left(\frac{s-\pi(X)}{2}\right)}\prod_{n=1}^{\infty}\left\{\det\left(I+\frac{s-\pi(X)}{2n}\right)e^{-\mathrm{trace}\left(\frac{s-\pi(X)}{2n}\right)}\right\}$$

where γ is Euler's constant. The function $\Gamma(s,\,\pi,\phi)$ can be expressed as a product of Γ-functions. Set

$$\xi(s,\,\pi,\,\phi) = \Gamma(s,\,\pi,\phi)L(s,\pi,\,\phi).$$

The functional equation to expect is

$$\xi(s,\,\pi,\,\phi) = \xi(1-s,\,\widetilde{\pi},\,\phi)$$

if $\widetilde{\pi}$ is the representation contragredient to π.

3. SPHERICAL FUNCTIONS

Each g_p that occurs in the expression on the left of (2) can be chosen to lie in ${}^cT_{\mathbb{C}}$. To see that the product converges in a half-plane it would be enough to show that for all λ in ${}^cL$ and all p

$$|\xi_\lambda(g_p)| \le \bar{\lambda}(\rho)$$

where $\bar{\lambda}$ is that element in the orbit of λ under the Weyl group which lies in the positive Weyl chamber and ρ is one-half the sum of the positive roots. We can associate to each g in ${}^cT_{\mathbb{C}}$ a point $\mu = \mu(g)$ in ${}^c\mathfrak{h}_{\mathbb{C}}$ so that* $\xi_\lambda(g) = p^{\lambda(\mu)}$ for all λ in ${}^cL$. The point μ is not uniquely determined by g but its real part is. If $\mu_p = \mu(g_p)$ we have to show that

$$\mathrm{Re}\ \lambda(\mu_p) \le \bar{\lambda}(\rho)\,.$$

The class $\{g_p\}$ is associated to the homomorphism χ_p of H_p into $\mathbb{C}$ determined by ϕ. This homomorphism has the property, not shared by all homomorphisms, that

$$|\chi_p(f)| \le c \int\limits_{G_{\mathbb{Q}_p}} |f(g)|\,dg$$

for all f in H_p. The factor c is a fixed constant. To prove this we must recall that ϕ is a cusp form and hence bounded. If M is a bound for ϕ and if $\phi(g_0) \ne 0$ then

$$|\chi_p(f)\phi(g_0)| = \left| \int\limits_{G_{\mathbb{Q}_p}} \phi(g_0 h) f(h)\,dh \right| \le M \int\limits_{G_{\mathbb{Q}_p}} |f(h)|\,dh$$

and the assertion follows.

Lemma. *Suppose μ in ${}^ch_{\mathbb{C}}$ is associated to the homomorphism χ_μ of H_p into $\mathbb{C}$. If there is a constant c such that*

$$|\chi_\mu(f)| \le c \int\limits_{G_{\mathbb{Q}_p}} |f(g)|\,dg$$

for all f, then

$$\mathrm{Re}\ \lambda(\mu) \le \bar{\lambda}(\rho)$$

for all λ.

If w is the homomorphism defined by $w(t) = p^{\lambda(t)(\mu)}$, set

$$\psi_\mu(g) = \psi_w(g)\,.$$

By definition

$$\chi_\mu(f)\psi_\mu(g) = \int\limits_{G_{\mathbb{Q}_p}} \psi_\mu(gh) f(h)\,dh\,.$$

If

$$\phi_\mu(g) = \int\limits_{G_{\mathbb{Z}_p}} \psi_\mu(kg)\,dk\,,$$

* At the infinite prime we would take $\mu = X$ with X as above.

then $\phi_\mu(1) = 1$, and $\phi_\mu(k_1 g k_2) = \phi_\mu(g)$ if k_1 and k_2 lie in $G_{\mathbb{Z}_p}$. Moreover it is easily verified that

$$\chi_\mu(f)\phi_\mu(g) = \int\limits_{G_{\mathbb{Q}_p}} \phi_\mu(gh) f(h)\, dh\,.$$

If μ satisfies the assumption of the lemma, take f_g to be the characteristic function of the double coset $G_{\mathbb{Z}_p} g G_{\mathbb{Z}_p}$. Then $\chi_\mu(f_g)$ is equal to $\chi_\mu(f_g)\phi_\mu(1) = \phi_\mu(g)$ times the measure of $(G_{\mathbb{Z}_p} g G_{\mathbb{Z}_p})$. It follows immediately that $|\phi_\mu(g)| \le c$ for all g.

To prove the lemma, it will be necessary to study the asymptotic behavior of ϕ_μ for general values of μ. Let $T^-_{\mathbb{Q}_p}$ be the set of t in $T_{\mathbb{Q}_p}$ for which $-\lambda(t)$ lies in the positive Weyl chamber. Since

$$G_{\mathbb{Q}_p} = G_{\mathbb{Z}_p} T^-_{\mathbb{Q}_p} G_{\mathbb{Z}_p}$$

it is sufficient to study the function ϕ_μ on $T^-_{\mathbb{Q}_p}$.

So far we are free to choose Haar measures in any manner we like. We so choose them on $V_{\mathbb{Q}_p}$, $T_{\mathbb{Q}_p}$ and $G_{\mathbb{Z}_p}$ that the total measures of $V_{\mathbb{Z}_p}$, $T_{\mathbb{Z}_p}$ and $G_{\mathbb{Z}_p}$ are 1. Then we choose the Haar measure on $G_{\mathbb{Q}_p}$ so that

$$\int\limits_{G_{\mathbb{Q}_p}} f(g)\, dg = \int\limits_{V_{\mathbb{Q}_p}} \int\limits_{T_{\mathbb{Q}_p}} \int\limits_{G_{\mathbb{Z}_p}} f(vtk) p^{-2\lambda(t)(\rho)}\, dv\, dt\, dk\,.$$

Choose an f in H_p and let C be a compact set in $V_{\mathbb{Q}_p}$ such that the support of f is contained in $CT_{\mathbb{Q}_p}G_{\mathbb{Z}_p}$. There is a constant $c_f > 0$ such that if $|\xi_\alpha(t)| \le c_f$ for $\alpha > 0$, then

$$tCt^{-1} \subseteq G_{\mathbb{Z}_p}\,.$$

Choose a t satisfying this condition. Then

$$\begin{aligned}
\chi_\mu(f)\phi_\mu(t) &= \int\limits_{V_{\mathbb{Q}_p}} \int\limits_{T_{\mathbb{Q}_p}} \phi_\mu(tvs) f(vs) p^{-2\lambda(s)(\rho)}\, dv\, ds \\
&= \int\limits_{C} \int\limits_{T_{\mathbb{Q}_p}} \phi_\mu(tvt^{-1}ts) f(vs) p^{-2\lambda(s)(\rho)} dv\, ds \\
&= \int\limits_{C} \int\limits_{T_{\mathbb{Q}_p}} \phi_\mu(ts) f(vs) p^{-2\lambda(s)(\rho)} dv\ ds\,.
\end{aligned}$$

Set $\widetilde{\phi}_\mu(t) = \phi_\mu(t) p^{-\lambda(t)(\rho)}$. Replacing the integral over C by an integral over $V_{\mathbb{Q}_p}$ we see that

$$\chi_\mu(f)\widetilde{\phi}_\mu(t) = \int\limits_{T_{\mathbb{Q}_p}} \widetilde{\phi}_\mu(ts)\widetilde{f}(s)\, ds = \int\limits_{T} \widetilde{\phi}_\mu(ts)\widetilde{f}(s)\, ds$$

if $f \to \widetilde{f}$ is the Satake homomorphism of H_p into the group algebra H'_p of $\overline{T}$ defined by

$$\widetilde{f}(s) = p^{-\lambda(s)(\rho)} \int\limits_{V_{\mathbb{Q}_p}} f(vs)\, dv\,.$$

If ψ_1 and ψ_2 are two functions on $\overline{T}$ and $c > 0$, we shall say that $\psi_1 \simeq_c \psi_2$ if $\psi_1(t) = \psi_2(t)$ whenever $|\xi_\alpha(t)| \le c$ for all positive roots α. The set of equivalence classes forms a vector space W_c. Let W_∞ be the injective limit of the spaces W_c. If f lies in H'_p and ψ is a function on $\overline{T}$, then we define $\lambda(f)\psi$ by

$$\lambda(f)\psi(t) = \int\limits_T \psi(ts)f(s)\,ds\,.$$

We can also regard $\lambda(f)$ as an operator on W_∞. If ϕ_μ is the image of $\widetilde{\phi}_\mu$ in W_∞, then $\lambda(\widetilde{f})\phi_\mu = \chi_\mu(f)\phi_\mu$ for all f in H_p. Since H'_p is a finite module over the image of H_p, the space

$$W = \{\lambda(f)\Phi_\mu \mid f \in H'_p\}$$

is a finite-dimensional subspace of W_∞.

Choose $t_1, \dots, t_n$ such that

$$|\xi_{\alpha_i}(t_j)| = p^{-\delta_{ij}}\,.$$

The points $\lambda(t_1), \dots, \lambda(t_n)$ are a basis of ${}^cL$ over Z. Let δ_i be the characteristic function of $\{\bar{t}_i\}$ and let S_i and N_i be respectively the semisimple and nilpotent parts of the restriction of $\lambda(\delta_i)$ to W. The matrices S_i, N_i, where $1 \le i \le n$, all commute. Choose a basis $\Psi_1, \dots, \Psi_1$ of W with respect to which $S_1, \dots, S_n$ are in diagonal form. Let

$$S_i\psi_j = p^{\gamma_{ij}}\psi_j\,.$$

If a_i is the smallest integer satisfying $N_i^{a_i+1} = 0$, then

$$(S_i + N_i)^{l_i} = \sum_{r_i=0}^{a_i} \binom{l_i}{r_i} S_i^{l_i - r_i} N_i^{r_i}\,.$$

Let $\psi_1, \dots, \psi_l$ be representatives of $\Psi_1, \dots, \Psi_l$, and choose $c_0 > 0$ such that

$$\psi_i(tt_k) = \sum_j \big(s_k^{ji} + n_k^{ji}\big)\psi_j(t)$$

if $|\xi_\alpha(t)| \le c_0$ for α positive. Choose t_0 such that $|\xi_\alpha(t_0)| \le c_0$ if α is positive. If

$$\Omega_0 = \big(\psi_1(t_0), \dots, \psi_l(t_0)\big), \qquad \Omega(t) = \big(\psi_l(t), \dots, \psi_l(t)\big)\,,$$

and $l_1 \ge 0, \dots, l_n \ge 0$ then

$$\begin{aligned}
\Omega\left(t_0 \prod_{k=1}^n t_k^{l_k}\right) &= \Omega_0 \prod_{k=1}^n (S_k + N_k)^{l_k} \\
&= \Omega_0 \left\{ \sum_{r_1=0}^{a_1} \cdots \sum_{r_n=0}^{a_r} \binom{l_1}{r_1} \cdots \binom{l_n}{r_n} S_1^{-r_1} \dots S_n^{-r_n} N_1^{r_1} \cdots N_n^{r_n} \right\} \prod_{k=1}^n S_k^{l_k} \\
&= \Theta(l_1, \dots, l_n) \prod_{k=1}^n S_k^{l_k}
\end{aligned}$$

where $\Theta(l_1, \dots, l_n)$ is a row vector with entries which are polynomials in $l_1, \dots, l_n$. Choose μ_j such that $\lambda(t_i)(\mu_j) = \gamma_{ij}$, for $1 \le i \le n$. If $|\xi_\alpha(t)| \le |\xi_\alpha(t_0)|$ for α positive, then

$$\psi_j(t) = p^{\lambda(t)(\mu_j)}\theta_j\big(\lambda(t)\big)$$

where $\theta_j\big(\lambda(t)\big)$ is a polynomial in $\lambda(t)$. Thus there is a constant c_1 and polynomials $\xi_1\big(\lambda(t)\big), \dots, \xi_1\big(\lambda(t)\big)$ such that

$$\widetilde{\phi}_\mu(t) \simeq_{c_1} \sum_{j=1}^{1} p^{\lambda(t)(\mu_j)} \xi_j\big(\lambda(t)\big)\,.$$

For our purposes it will be necessary to know the relation between $\mu_1, \dots, \mu_1$ and μ and to have a more or less explicit formula for the polynomials $\xi_j\big(\lambda(t)\big)$. Let ${}^c\Omega$ be the Weyl group of ${}^cT$. For notational convenience let us use the map $\sigma \to {}^c\sigma$ to identify ${}^c\Omega$ and Ω, the Weyl group of T. Let

$$A = \{\mu \in {}^c h_{\mathbb{R}} \mid \sigma\mu = t\mu,\, \sigma,\, t \in \Omega \text{ implies } \sigma = t\}\,.$$

Let $\bar{t}_i$ be the image of t_i in $\overline{T}$ and let

$$\bar{t}_i^{(1)}, \dots, \bar{t}_i^{(b_i)}$$

be the distinct elements in the orbit of $\bar{t}_i$ under Ω. Set

$$B = \{\mu \in {}^c\mathfrak{h}_{\mathbb{R}} \mid \lambda(\bar{t}_i^{(k)})(\mu) = \lambda\big(\bar{t}_j^{(1)}\big)(\mu) \text{ implies } i = j \text{ and } k = 1\}\,.$$

Define $\mu_{k_1}, \dots, k_n$ by the condition that

$$\lambda(\bar{t}_i^{(k_j)})(\mu) = \lambda(\bar{t}_i)(\mu_{k_1}, \dots, k_n) \; 1 \le i \le n$$

and set

$$C = \{\mu \in {}^c\mathfrak{h}_{\mathbb{R}} \mid \sigma\mu = \mu_{k_1 \dots k_n} \text{ implies } \sigma\bar{t}_i^{(k_i)} = \bar{t}_i,\, 1 \le i \le n\}\,.$$

The complements of A, B, and C are the union of a finite number of proper affine subspaces of ${}^c\mathfrak{h}_{\mathbb{R}}$. Thus there is a point μ^0 in $A \cap B \cap C$. Choose t_0 such that

(i) $\lambda(t_0)(\sigma\mu^0) = (t_0)(\tau\mu^0)$ implies $\sigma = \tau$;

(ii) $\lambda(t_0)({}^c\sigma\mu^0) = (t_0)(\mu^0_{k_1,\dots,k_n})$ implies $\sigma\bar{t}_i^{(k_i)} = \bar{t}_i,\, 1 \le i \le n$.

Let S be the collection of points μ in ${}^c\mathfrak{h}_{\mathbb{C}}$ satisfying

(i) $p^{\lambda(t_0)(\sigma\mu)} = p^{\lambda(t_0)(\tau\mu)}$ implies $\sigma = \tau$;

(ii) $p^{\lambda(t_0)(\sigma\mu)} = p^{\lambda(t_0)(\mu_{k_1,\dots,k_n})}$ implies $\sigma\bar{t}_i^{(k_i)} = \bar{t}_i,\, 1 \le i \le n$;

(iii) $p^{\lambda(\bar{t}_i^{(k)})(\mu)} = p^{\lambda(\bar{t}_j^{(l)})(\mu)}$ implies $i = j$ and $k = l$.

Then S is an open, dense, and connected subset of ${}^c\mathfrak{h}_{\mathbb{C}}$.

Suppose μ lies in S. Since the coefficients of the polynomial

$$p_i(X) = \prod_{j=1}^{b_j} \big(X - \delta\big(\bar{t}_i^{(j)}\big)\big)$$

lie in the image of H_p, the equation

$$\lambda\big(p_i(X)\big)\Phi_\mu = \prod_{j=1}^{b_j} \big(X - p^{\lambda(t_i^{(j)})(\mu)}\big)\Phi_\mu$$

is satisfied. It is satisfied not only by Φ_μ but by every element of W. Since $p_i\big(\delta(\bar{t}_i)\big) = 0$, the minimal polynomial of the restriction of $\lambda\big(\delta(\bar{t}_i)\big)$ to W divides

$$\prod_{j=1}^{b_j} \big(X - p^{\lambda(\bar{t}_i^{(j)})(\mu)}\big)\,.$$

Since this polynomial has no multiple root, $N_i = 0$ and

$$\lambda(\delta_t)\Psi_j = p^{\lambda(t)(\mu_j)}\Psi_j$$

for all t. Here δ_t is the characteristic function of $\{\bar{t}\}$. The point μ_j must be equivalent modulo $2\pi i L/\log p$ to an element in the orbit, under Ω, of μ. If not, there is a t such that

$$p^{\lambda(t)(\sigma\mu)} \neq p^{\lambda(t)(\mu_j)}$$

for any σ in Ω. This is impossible because the minimal polynomial of the restriction of $\lambda(\delta_t)$ is divisible by

$$X - p^{\lambda(t)(\mu_j)}$$

and divides

$$\prod_\sigma \left(X - p^{\lambda(t)(\sigma\mu)}\right).$$

Thus there is a constant c_μ and constants $a_\sigma(\mu)$ such that

$$\widetilde{\phi}_\mu(t) \simeq_{c_\mu} \sum_\sigma a_\sigma(\mu) p^{\lambda(t)(\sigma\mu)}.$$

We need to prove also that the constant c_μ can be chosen to be independent of μ. Since the coefficients of $p_i(X)$ are independent of μ, there is a constant c_1 such that

$$\lambda\bigl(p_i(X)\bigr)\widetilde{\phi}_\mu \simeq_{c_1} \prod_{j=1}^{b_i} \bigl(X - p^{\lambda(\bar{t}_i^{(j)})(\mu)}\bigr)\widetilde{\phi}_\mu$$

for all μ. If

$$\widetilde{\phi}_\mu(\,\bullet\,, k_1, \ldots, k_n) = \prod_{i=1}^n \Biggl\{ \prod_{\substack{j_i \neq k_i \\ 1 \leq j_i \leq b_i}} \left(\frac{\lambda(\delta_i) - p^{\lambda\left(\bar{t}_i^{(j_i)}\right)(\mu)}}{p^{\lambda\left(t_i^{(k_i)}\right)(\mu)} - p^{\lambda\left(t_i^{(j_i)}\right)(\mu)}} \right) \Biggr\} \widetilde{\phi}_\mu$$

then

$$\widetilde{\phi}_\mu = \sum_{k_1=1}^{b_1} \cdots \sum_{k_n=1}^{b_n} \widetilde{\phi}_\mu(\,\bullet\,, k_1, \ldots, k_n)$$

and, for some constant c_2 which does not depend on μ,

$$\lambda(\delta_i)\widetilde{\phi}_\mu(\,\bullet\,, k_1, \ldots, k_n) \simeq_{c_2} p^{\lambda(t_i)(\mu_{k_1,\ldots,k_n})}\widetilde{\phi}_\mu,\ 1 \leq i \leq n.$$

If

$$p_0(X) = \prod_\sigma \bigl(X - p^{\lambda(t_0)(\sigma\mu)}\bigr)$$

there is a constant c_3 such that

$$p_0\bigl(\lambda(\delta_{t_0})\bigr)\widetilde{\phi}_\mu(\,\bullet\,, k, \ldots, k_n) \simeq_{c_3} 0$$

and

$$p_0\bigl(\lambda(\delta_{t_0})\bigr)\widetilde{\phi}_\mu(\,\bullet\,, k_1, \ldots, k_n) \simeq_{c_3} p_0(p^{\lambda(t_0)(\mu_{k_1,\ldots,k_n})})\widetilde{\phi}_\mu(\,\bullet\,, k_1, \ldots, k_n).$$

Since μ lies in S.

$$\widetilde{\phi}_\mu(\,\bullet\,, k_1, \ldots, k_n) \simeq_{c_3} 0$$

unless $\sigma\bar{t}_i^{k_i} = \bar{t}_i$, $1 \le i \le n$, for some σ in Ω. Then $\mu_{k_1,\dots,k_n} = \sigma\mu$. For a given σ, let $k_1(\sigma),\dots,k_n(\sigma)$ be the indices satisfying $\sigma\bar{t}_i^{(k_i(\sigma))} = \bar{t}_i$ and set

$$\widetilde{\phi}_\mu(\,\bullet\,,\sigma) = \widetilde{\phi}_\mu\big(\,\bullet\,, k_1(\sigma),\dots,k_n(\sigma)\big).$$

There is a constant c_4 such that

$$\widetilde{\phi}_\mu \simeq_{c_4} \sum_\sigma \widetilde{\phi}_\mu(\,\bullet\,,\sigma)$$

and

$$\widetilde{\phi}_\mu(\cdot,\sigma) \simeq_{c_4} \; a_\sigma(\mu)p^{\lambda(t)(\sigma\mu)}$$

for all μ in S.

The next step is to evaluate the coefficients $a_\sigma(\mu)$. Since $\phi_{\sigma\mu} = \phi_\mu$, the same is true of $\widetilde{\phi}_\mu$. If μ lies in $\cap_\sigma \sigma S$ which is an open, dense, and connected set, then

$$\sum_\tau a_\tau(\mu)p^{\lambda(t)(\mu)} \simeq_{c_4} \sum_\tau a_\tau(\sigma\mu)p^{\lambda(t)(\tau\sigma\mu)}.$$

As a consequence $a_{\tau\sigma}(\mu) = a_\tau(\sigma\mu)$. Thus it is enough to evaluate $a(\mu) = a_{\sigma_0}(\mu)$ if σ_0 is the element of the Weyl group which takes every positive root to a negative root. Since $a(\mu)$ is an analytic function, it is enough to evaluate it when Re μ lies in the interior of the positive Weyl chamber.

Suppose $\lambda(t)$ lies in the interior of the positive Weyl chamber. Since

$$\phi_\mu(t^n) = \phi_\mu\big((\sigma_0 t)^n\big) = \widetilde{\phi}_\mu\big((\sigma_0 t)^n\big)p^{-n\lambda(t)(\rho)}$$

the relation

$$\phi_\mu(t^n) = \sum_\sigma a_\sigma(\mu)p^{n\lambda(t)(\sigma_0^{-1}\sigma\mu-\rho)}$$

is valid for sufficiently large n and

$$\lim_{n\to\infty} p^{-n\lambda(t)(\mu-\rho)}\phi_\mu(t^n) = a_{\sigma_0}(\mu) = a(\mu)$$

because Re $\lambda(t)(\mu - \sigma\mu) > 0$.

We shall evaluate this limit in another way and obtain $a(\mu)$. Recall that

$$\phi_\mu(t) = \int\limits_{G_{\mathbb{Z}_p}} \psi_\mu(kt)\,dk\,.$$

Following Harish-Chandra we study this integral by means of the following easily proved lemma.

Lemma. *Suppose $\overline{V}$ is the unipotent radical of the parabolic group opposed to B. If $\bar{v}$ lies in $\overline{V}_{\mathbb{Q}_p}$, let $\bar{v} = v(\bar{v})t(\bar{v})k(\bar{v})$ with $v(\bar{v})$ in $V_{\mathbb{Q}_p}$, $t(\bar{v})$ in $T_{\mathbb{Q}_p}$, and $k(\bar{v})$ in $G_{\mathbb{Z}_p}$. Set $\lambda(\bar{v}) = \lambda\big(t(\bar{v})\big)$. There is a constant a such that if ψ is any integrable function on $B_{\mathbb{Z}_p}\backslash G_{\mathbb{Z}_p}$ then*

$$a\int\limits_{G_{\mathbb{Z}_p}} \psi(k)\,dk = \int\limits_{\overline{V}_{\mathbb{Q}_p}} \psi\big(k(\bar{v})\big)p^{\lambda(\bar{v})(2\rho)}\,d\bar{v}\,.$$

I ask you to bear in mind for a while that a must necessarily equal

$$\int\limits_{\overline{V}_{\mathbb{Q}_p}} p^{\lambda(\bar{v})(2\rho)}\,d\bar{v}.$$

Using the lemma we see that

$$\begin{aligned}
\phi_\mu(t) &= \frac{1}{a}\int\limits_{\overline{V}_{\mathbb{Q}_p}} \psi_\mu\big(t^{-1}(\bar v)\bar v t\big) p^{\lambda(\bar v)(2\rho)}\,d\bar v \\
&= \frac{1}{a}\int\limits_{\overline{V}_{\mathbb{Q}_p}} \psi_\mu(\bar v t) p^{-\lambda(\bar v)(\mu-\rho)}\,d\bar v \\
&= \frac{p^{\lambda(t)(\mu+\rho)}}{a}\int\limits_{V_{\mathbb{Q}_p}} \psi_\mu(t^{-1}\bar v t) p^{-\lambda(\bar v)(\mu-\rho)}\,d\bar v \\
&= \frac{p^{\lambda(t)(\mu-\rho)}}{a}\int\limits_{\overline{V}_{\mathbb{Q}_p}} \psi_\mu(\bar v) p^{-\lambda(t\bar v t^{-1})(\mu-\rho)}\,d\bar v \\
&= \frac{p^{\lambda(t)(\mu-\rho)}}{a}\int\limits_{V_{\mathbb{Q}_p}} p^{\lambda(\bar v)(\mu+\rho)} p^{-\lambda(t\bar v t^{-1})(\mu-\rho)}\,d\bar v\,.
\end{aligned}$$

Thus

(3)
$$a(\mu) = \frac{1}{a}\lim_{n\to\infty}\int\limits_{\overline{V}_{\mathbb{Q}_p}} p^{\lambda(\bar v)(\mu+\rho)} p^{-\lambda(t^n\bar v t^{-n})(\mu-\rho)}\,d\bar v$$

if t lies in the interior of the positive Weyl chamber.

Lemma. *If ν lies in the positive Weyl chamber, if $\lambda(t)$ does also, and if $\bar v$ lies in $\overline{V}_{\mathbb{Q}_p}$ then $\lambda(\bar v)(\nu) \le 0$ and $\lambda(\bar v)(\nu) \le \lambda(t\bar v t^{-1})(\nu)$.*

If g lies in $G_{\mathbb{Q}_p}$ and $g = vsk$, with v in $T_{\mathbb{Q}_p}$, s in $T_{\mathbb{Q}_p}$, and k in $G_{\mathbb{Z}_p}$ set $\lambda(g) = \lambda(s)$. It is known that if t satisfies the condition of the lemma, then

$$\lambda(kt)(\nu) \le \lambda(t)(\nu)\,.$$

Since

$$\lambda(gt) = \lambda(g) + \lambda(kt)\,,$$

we have

$$\lambda(g)(\nu) + \lambda(t)(\nu) \ge \lambda(gt)(\nu)\,.$$

Moreover

$$\lambda(t^{-1}gt)(\nu) = -\lambda(t)(\nu) + \lambda(gt)(\nu) \le \lambda(g)(\nu)\,.$$

The second assertion of the lemma follows. If $\bar v$ lies in $\overline{V}_{\mathbb{Q}_p}$, there is a t with $\lambda(t)$ in the positive Weyl chamber such that $t\bar v t^{-1}$ lies in $G_{\mathbb{Z}_p}$. Since $\lambda(tvt^{-1})$ is then zero, the first assertion follows from the second.

If R is any open half-space in ${}^c\mathfrak{h}_{\mathbb{R}}$ which is bounded by a hyperplane passing through zero and if $\bar R$ is its closure, let $\sum_R$ be the set of roots lying in R, let $\sum_R^+$ be the set of positive roots lying in $\bar R$, and let $\sum_R^-$ be the set of negative roots lying in R. Let $\bar n(R)$ be the Lie algebra spanned by the root vectors corresponding to roots in $\sum_R^-$ and let $\overline{N}(R)$ be the group with Lie algebra $\bar n(R)$. If p is any prime, finite or infinite, we consider

$$\int\limits_{\overline{N}_{\mathbb{Q}_p}(R)} \psi_\mu(\bar n)\,d\bar n = \delta_R(\mu)\,.$$

It has been shown by Gindikin and Karpelevich that, when $p = \infty$, this integral converges if $\operatorname{Re}\mu(H_\alpha) > 0$ for every positive root α and is equal to

$$\prod_{-\alpha\in\sum_R^-} \frac{\pi^{1/2}\,\Gamma\left(\frac{\mu(H_\alpha)}{2}\right)}{\Gamma\left(\frac{1+\mu(H_\alpha)}{2}\right)}$$

If X_α are the root vectors belonging to the Chevalley basis, the Haar measure is that associated to the form which takes the value 1 on

$$\bigwedge_{\alpha\in\sum_R^-} X_\alpha\,.$$

We shall imitate their proof and show that the integral converges in the same region when p is finite and is equal to

$$\prod_{-\alpha\in\sum_R^-} \frac{1-\dfrac{1}{p^{\mu(H_\alpha)+1}}}{1-\dfrac{1}{p^{\mu(H_\alpha)}}}. \tag{4}$$

For the moment we shall assume this and complete our evaluation of the limit (3). Choose $\epsilon > 0$ such that $\operatorname{Re}(\mu) - \epsilon\rho$ lies in the positive Weyl chamber. Then

$$\operatorname{Re}\{\lambda(\bar v)(\mu+\rho) - \lambda(t^n\bar v t^{-n})(\mu-\rho)\}$$

is the sum of

$$\operatorname{Re}\{\big(\lambda(\bar v) - \lambda(t^n\bar v t^{-n})\big)(\mu-\epsilon\rho) + \lambda(t^n\bar v t^{-n})(\rho-\epsilon\rho)\}$$

and

$$\operatorname{Re}\{\lambda(\bar v)(\rho+\epsilon\rho)\}\,.$$

It follows from the lemma that the first expression is less than or equal to zero. Since

$$\int_{\overline{V}_{\mathbb{Q}_p}} p^{\operatorname{Re}\{\lambda(\bar v)(\rho+\epsilon\rho)\}}\,d\bar v$$

is finite, we can take the limit under the integral sign in (3) to obtain

$$\begin{aligned} a(\mu) &= \frac{1}{a}\int_{\overline{V}_{\mathbb{Q}_p}} p^{\lambda(\bar v)(\mu+\rho)}\,d\bar v \\ &= \frac{1}{a}\prod_{\alpha>0} \frac{1-\dfrac{1}{p^{\mu(H_\alpha)+1}}}{1-\dfrac{1}{p^{\mu(H_\alpha)}}} \end{aligned}$$

Thus there is a constant c such that if $|\xi_\alpha(t)| \ge c$ for α positive then

$$\phi_\mu(t) = \frac{1}{a}\sum_\sigma \left\{\prod_{\alpha>0} \frac{1-\dfrac{1}{p^{\sigma\mu(H_\alpha)+1}}}{1-\dfrac{1}{p^{\sigma\mu(H_\alpha)}}}\right\} p^{\lambda(t)(\sigma\mu-\rho)}\,. \tag{5}$$

Let

$$\prod_{\alpha>0}\left(1-\frac{1}{p^{\nu(H_\alpha)+1}}\right)=\sum_{\bar{s}\in\overline{T}} b_{\bar{s}}p^{\lambda(\bar{s})(\nu)}\,.$$

Only a finite number of the coefficients are not zero. If ${}^c\alpha$ is the root of ${}^c\mathfrak{h}$ corresponding to α the formula (5) may be written

$$\phi_\mu(t)=\frac{1}{a}\sum_{\bar{S}} b_{\bar{s}}\left\{\frac{\sum_\sigma \operatorname{sgn}\sigma\, p^{(\lambda(\bar{t}\bar{s})+{}^c\rho)(\sigma\mu)}}{\prod_{{}^c\alpha>0}\left(p^{\frac{{}^c\alpha(\mu)}{2}}-p^{-\frac{{}^c\alpha(\mu)}{2}}\right)}\right\}p^{-\lambda(t)\rho}. \tag{6}$$

This formula is valid for all μ. The relation of this formula to the Weyl character formula need not be pointed out to the knowledgeable reader.

I do not know if it is valid for all t. However it is valid for $t=1$. To prove this we show that the right side is 1 when $t=1$. First of all, it follows from the formulae for $\delta_R(\mu)$ that

$$a=\prod_{\alpha>0}\frac{1-\dfrac{1}{p^{\rho(H_\alpha)+1}}}{1-\dfrac{1}{p^{\rho(H_\alpha)}}}=\prod_{{}^c\alpha>0}\frac{1-\dfrac{1}{p^{{}^c\alpha(\rho)+1}}}{1-\dfrac{1}{p^{{}^c\alpha(\rho)}}}.$$

Now $b_{\bar{s}}$ is zero unless

$$\lambda(\bar{s})=\sum_{{}^c\alpha\in w}{}^c\alpha$$

where w is a subset of the positive roots of ${}^c\mathfrak{h}$. Then ${}^c\rho+\lambda(\bar{s})$ is either singular or in the orbit of ${}^c\rho$ under ${}^c\Omega$. To prove this* we recall that Kostant has shown in lemma 5.9 of his paper on the Borel-Weil theorem that every element in the orbit of ${}^c\rho+\lambda(\bar{s})$ is of the form ${}^c\rho+\lambda(\bar{s}')$ with

$$\lambda(\bar{s}')=-\sum_{{}^c\alpha\in w'}^{c}\alpha$$

and suppose that ${}^c\rho+\lambda(\bar{s})$ lies in the positive Weyl chamber. If it is nonsingular, it equals ${}^c\rho+\lambda$ with λ in the positive Weyl chamber. Then $\lambda=\lambda(\bar{s})$. It follows immediately that $\lambda=\lambda(\bar{s})=0$. If $b_{\bar{s}}$ is not zero, the corresponding term in brackets on the right side of (5) is zero when $t=1$ and $\lambda(\bar{s})+{}^c\rho$ is singular and is ±1 when $t=1$ and $\lambda(s)+{}^c\rho$ is in the orbit of ${}^c\rho$. In any case if, for brevity, we denote the right side of (6) by $\Theta_\mu(t)$, then $\Theta_\mu(1)$ is independent of μ. Thus

$$\Theta_\mu(1)=\Theta_\rho(1)=\frac{1}{a}\sum_\sigma\left\{\prod_{\alpha>0}\frac{1-\dfrac{1}{p^{\sigma\rho(H_\alpha)+1}}}{1-\dfrac{1}{p^{\sigma\rho(H_\alpha)}}}\right\}.$$

Suppose $\sigma\neq1$. Then, for some simple root $\alpha_0,\sigma\alpha_0=-\beta_0$ is negative and

$$\sigma\rho(H_{\beta_0})=\rho(H_{\alpha_0})=-1$$

and the corresponding term in the above sum is zero. Thus

$$\Theta_\mu(1)=\frac{1}{a}\left\{\prod_{\alpha>0}\frac{1-\dfrac{1}{p^{\rho(H_\alpha)+1}}}{1-\dfrac{1}{p^{\rho(H_\alpha)}}}\right\}=1\,.$$

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* [Added 1970] I now notice that I made the matter unnecessarily complicated.

Since $\Theta_\mu(t)$ is a linear combination of products of exponentials and polynomials in $\lambda(t)$ it cannot vanish in an open cone without vanishing identically. This the last formula shows it cannot do.

We are now in a position to show that if ϕ_μ is bounded then

$$\mathrm{Re}\ \lambda(\mu) \le \bar{\lambda}(\rho)$$

for all λ. We may suppose that $\mathrm{Re}\ \mu$ lies in the positive Weyl chamber. Then

$$\mathrm{Re}\ \lambda(\mu) \le \mathrm{Re}\ \bar{\lambda}(\mu)\,,$$

and we need only consider λ lying in the positive Weyl chamber. It will be simpler to consider only λ lying in the interior of the positive Weil chamber. The assertion for points on the boundary can be obtained by a simple limiting argument. Then if $\mathrm{Re}\ (\sigma\mu) \ne \mathrm{Re}\ \mu$, for some σ in Ω, $\mathrm{Re}\ \ \lambda(\sigma\mu) < \mathrm{Re}\ \lambda(\mu)$.

Let w be the set of simple roots α for which $\mathrm{Re}\ {}^c\alpha(\mu) = 0$. Let $\sum_0^+(w)$ be the set of all positive roots which are linear combinations of the elements of w, and let $\sum^+(w)$ be the other positive roots. Let G_1 be the subgroup of G corresponding to the Lie algebra generated by the root vectors associated to the elements of $\sum^+(w)$ and their negatives and let $\Omega_1 \subseteq \Omega$ be the Weyl group of G_1. If σ belongs to Ω and $\mathrm{Re}\ \sigma(\mu) = \mathrm{Re}\ \mu$, then σ belongs to Ω_1. Set $\Theta'(t)$ equal to

$$\frac{1}{a}\sum_{\sigma\in\Omega_1}\left\{\prod_{\alpha\in\sum^+(w)}\frac{1-\dfrac{1}{p^{\sigma\nu(H_\alpha)+1}}}{1-\dfrac{1}{p^{\sigma\nu(H_\alpha)}}}\right\}\left\{\prod_{\alpha\in\sum_0^+(w)}\frac{1-\dfrac{1}{p^{\sigma\nu(H_\alpha)+1}}}{1-\dfrac{1}{p^{\sigma\nu(H_\alpha)}}}\right\}p^{\lambda(t)(\sigma\nu-\rho)}\,.$$

Since $\lambda(t) = \lambda_1 + \lambda_2$ with $\lambda_1 = \lambda(t_1)$ for some t_1 in the adjoint group of G_1 and $\lambda_2(\alpha) = 0$ for α in w, we can write $\Theta'(t)$ as the product of

$$\frac{1}{a}\left\{\prod_{\alpha\in\sum^+(w)}\frac{1-\dfrac{1}{p^{\nu(H_\alpha)+1}}}{1-\dfrac{1}{p^{\nu(H_\alpha)}}}\right\}$$

and

$$\left\{\sum_{\sigma\in\Omega_1}\left(\prod_{\alpha\in\sum_0^+(w)}\frac{1-\dfrac{1}{p^{\sigma\nu(H_\alpha)+1}}}{1-\dfrac{1}{p^{\sigma\nu(H_\alpha)}}}\right)p^{\lambda_1(\sigma\nu-\rho)}\right\}p^{\lambda_2(\nu-\rho)}\,.$$

Applying the previous discussion to G_1 instead of G, we see that Θ'_ν is analytic at μ. Moreover $\Theta'_\nu(t)$ does not vanish, as a function of t, for $\lambda(t)$ in an open cone. Set $\Theta''_\nu(t) = \Theta_\nu(t) - \Theta'_\nu(t)$; $\Theta''_\nu(t)$ is also analytic at μ. As a function of t, $\Theta''_\mu(t)$ is a linear combination of terms of the form $p\big(\lambda(t)\big)p^{\lambda(t)(\mu'-\rho)}$ where μ' is an element in the orbit of μ with $\mathrm{Re}\ \mu' \ne \mathrm{Re}\ \mu$ and $p\big(\lambda(t)\big)$ is a polynomial in $\lambda(t)$. Thus, if $\lambda(t)$ lies in the interior of the positive Weyl chamber,

$$\lim_{n\to\infty} p^{-n\lambda(t)(\mu-\rho)}\Theta''_\mu(t^n) = 0$$

and

$$\lim_{n\to\infty} p^{-n\lambda(t)(\mu-\rho)}\phi_\mu(t^n) = \lim_{n\to\infty} p^{-n\lambda(t)(\mu-\rho)}\Theta'_\mu(t^n)\,.$$

Suppose ϕ_μ were bounded and for some λ in the positive Weyl chamber $\mathrm{Re}\ \lambda(\rho-\mu)$ were less than zero. Then there would exist a t such that $\mathrm{Re}\ \lambda(t)(\rho-\mu) < 0$ and $\Theta'_\mu(t^n)$ did not vanish identically. Then

$$\lim_{n\to\infty} p^{-n\lambda(t)(\mu-\rho)}\Theta'_\mu(t^n) = 0\,.$$

On the other hand if t is fixed

$$p^{-n\lambda(t)(\mu-\rho)}\Theta'_\mu(t^n) = \sum_{k=0}^{q} \phi_k(n) n^k$$

where $\phi_k(m)$ is a linear combination of exponentials $e^{i\alpha m}$ with α real. We can suppose $\phi_q(n) \not\equiv 0$. Certainly

$$\lim_{n\to\infty} \phi_q(n) = 0\,.$$

Let

$$\phi_q(n) = \sum_{j=1}^{r} a_j e^{i\alpha_j n}$$

with $\alpha_1, \ldots, \alpha_r$ real and incongruent modulo 2π. Then

$$0 = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \phi_q(n) e^{-i\alpha_j n} = a_j\,.$$

This is a contradiction.

4. THE FORMULA OF GINDIKIN AND KARPELEVICH

To complete the proof of the lemma and to prepare ourselves for the next stage of the argument, we must evaluate the functions $\delta_R(\mu)$ in closed form. The argument is an induction on the number of elements in $\sum_R^-$. Since $\delta_R(\mu)$ is certainly 1 when $\sum_R^-$ is empty, we can start immediately with the induction step. Let C_R^- and C_R^+ be the convex cones with vertex at the origin generated by $\sum_R^-$ and $\sum_R^+$ respectively. Let

$$D_R^+ = \{\lambda \in \mathfrak{h}_R \mid \lambda(\mu) \geq 0 \text{ for all } \mu \in C_R^+\},$$
$$D_R^- = \{\lambda \in \mathfrak{h}_R \mid \lambda(\mu) \geq 0 \text{ for all } \mu \in C_R^-\}.$$

If, as before, ${}^c\rho = \frac{1}{2}\sum_{\alpha>0} {}^c\alpha$, then ${}^c\rho$ lies in the interior of D_R^+ and, if $\sum_R^-$ is not empty, in the exterior of D_R^-. If

$$R = \{\mu \mid \lambda_0(\mu) \geq 0\},$$

then λ_0 lies in the intersection of D_R^+ with the interior of D_R^-. Joining ${}^c\rho$ to λ_0, we pass through a point of the boundary of D_R^- which lies in the interior of D_R^+. Since D_R^- is polygonal, there is a point λ_1 near this boundary point which lies inside an $n-1$ dimensional side of D_R^- and in the interior of D_R^+. Then $\sum_R^-$ is the set of all negative roots satisfying $\lambda_1(\alpha) \geq 0$. There is exactly one negative root $-\alpha_0$ such that $\lambda_1(-\alpha_0) = 0$. Let

$$S = \{\mu \mid \lambda_1(\mu) > 0\}.$$

Then $\sum_R^-$ is the union of $-\alpha_0$ and $\sum_S^-$.

To establish the formula (4) we show that

$$\delta_R(\mu) = \frac{1 - \dfrac{1}{p^{\mu(H_{\alpha_0})+1}}}{1 - \dfrac{1}{p^{\mu(H_{\alpha_0})}}}\, \delta_S(\mu).$$

We shall also see that the integral defining $\delta_R(\mu)$ converges if that defining $\delta_S(\mu)$ does and

$$\operatorname{Re} \mu(H_{\alpha_0}) > 0.$$

Let $\overline{N}^0$ be the one parameter group generated by the root vector $X_{-\alpha_0}$ belonging to $-\alpha_0$. Let G^0 be the group corresponding to the Lie algebra spanned by $X_{\alpha_0},\ X_{-\alpha_0},$ and H_{α_0}. As usual there is a mapping of $SL(2)$ into G^0 such that the image of

$$\begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}$$

is $\exp xX_{-\alpha_0}$ and the image of

$$\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$$

is $\exp xX_{\alpha_0}$. The image of $SL(2, \mathbb{Z}_p)$ is contained in $G_{\mathbb{Z}_p}$. If $\bar{n}_1 = \exp xX_{-\alpha_0}$, let a_1 be the identity if x lies in $\mathbb{Z}_p$ and let a_1 be the image of

$$\begin{pmatrix} x^{-1} & 0 \\ 0 & x \end{pmatrix}$$

if x is not in $\mathbb{Z}_p$. Let n_1 be the identity if x is in $\mathbb{Z}_p$ and let n_1 be the image of

$$\begin{pmatrix} 1 & x^{-1} \\ 0 & 1 \end{pmatrix}$$

if x is not in $\mathbb{Z}_p$. In all cases $\bar{n}_1$ lies in $n_1 a_1 G_{\mathbb{Z}_p}$. Thus, if $\bar{n}_2$ lies in $\overline{N}_{\mathbb{Q}_p}(S)$,

$$\psi_\mu(\bar{n}_2\bar{n}_1) = \psi_\mu(\bar{n}_2 n_1 a_1) = \psi_\mu(a_1)\psi_\mu(a_1^{-1}n_1^{-1}n_2 n_1 a_1)\,.$$

Consequently

$$\begin{aligned}\int\limits_{\overline{N}_{\mathbb{Q}_p}(R)} \psi_\mu(\bar{n})\,d\bar{n} &= \int\limits_{\overline{N}^0_{\mathbb{Q}_p}} \Big\{ \int\limits_{\overline{N}_{\mathbb{Q}_p}(S)} \psi_\mu(\bar{n}_2\bar{n}_1)\,d\bar{n}_2 \Big\}\, d\bar{n}_1 \\ &= \int\limits_{\overline{N}^0_{\mathbb{Q}_p}} \psi_\mu(a_1)\Big\{ \int\limits_{\overline{N}_{\mathbb{Q}_p}(S)} \psi_\mu(a_1^{-1}n_1^{-1}\bar{n}_2 n_1 a_1) d\bar{n}_2 \Big\}\, d\bar{n}_1\,.\end{aligned}$$

Let

$$\mathfrak{n} = \sum_{\lambda_1(\alpha)>0} Q_p X_\alpha, \qquad \mathfrak{a} = \sum_{\substack{\lambda_1(\alpha)>0\\ \alpha>0}} Q_p X_\alpha, \qquad \mathfrak{b} = \sum_{\substack{\lambda_1(\alpha)>0\\ \alpha<0}} Q_p X_\alpha\,.$$

Here $\mathfrak{n}$ is the direct sum of $\mathfrak{a}$ and $\mathfrak{b}$. If Q is a closed half-space contained in S, let Θ_Q be the set of roots contained in Q. Let the distinct collections of roots obtained in this way be, in decreasing order, $\Theta_0, \Theta_1, \ldots, \Theta_l, \Theta_{l+1} = \phi$, and set

$$\mathfrak{n}_k = \sum_{\alpha\in\Theta_k} Q_p X_\alpha\,.$$

The relations $[\mathfrak{n},\, \mathfrak{n}_k] \subseteq \mathfrak{n}_{k+1}$ and $\mathfrak{n}_k = \mathfrak{a}\cap\mathfrak{n}_k + \mathfrak{b}\cap\mathfrak{n}_k$ are clear; in particular, $\mathfrak{n}$ is nilpotent. The following rather complicated lemma is an easy consequence of the Campbell-Hausdorff formula.

Lemma. *Suppose $\mathfrak{n}$ is a Lie algebra of nilpotent transformations of a vector space V over a field k of characteristic zero and N is the associated group of linear transformations. Suppose*

$$\mathfrak{n} = \mathfrak{n}_0 \supsetneqq \mathfrak{n}_1 \supsetneqq \ldots \supsetneqq \mathfrak{n}_{l+1} = \{0\}$$

is a decreasing sequence of ideals in $\mathfrak{n}$ and $[\mathfrak{n}, \mathfrak{n}_k] \subseteq \mathfrak{n}_{k+1}$. Suppose that $\mathfrak{a}$ and $\mathfrak{b}$ are two subspaces of $\mathfrak{n}$ and

$$\mathfrak{n}_k = \mathfrak{n}_k \cap \mathfrak{a} \oplus\, \mathfrak{n}_k \cap \mathfrak{b}$$

for each k. Set $\mathfrak{a}_k = \mathfrak{n}_k\cap\mathfrak{a}$, $\mathfrak{b}_k = \mathfrak{n}_k\cap\mathfrak{b}$, and choose $\widetilde{\mathfrak{a}}_i$, $\widetilde{\mathfrak{b}}_i$ such that

$$\mathfrak{a}_k = \sum_{i=k}^{l} \oplus\, \widetilde{\mathfrak{a}}_i \text{ and } \mathfrak{b}_k = \sum_{i=k}^{l} \oplus\, \widetilde{\mathfrak{b}}_i\,.$$

Then every element of N can be written uniquely as

$$n = \exp X_0 \exp X_1 \ldots \exp X_1 \exp Y_1 \ldots \exp Y_0$$

with X_i in $\widetilde{\mathfrak{a}}_i$, Y_i in $\widetilde{\mathfrak{b}}_i$. If $X \to X^a$ is an automorphism of $\mathfrak{n}$ leaving each $\mathfrak{n}_k$ invariant let $X_k + Y_k \to X'_k + Y'_k$ be the induced transformation on

$$\mathfrak{n}_k/\mathfrak{n}_{k+1} \simeq \widetilde{\mathfrak{a}}_k \oplus\, \widetilde{\mathfrak{b}}_k.$$

If

$$n^a = \exp X''_0 \ldots \exp X''_1 \exp Y''_1 \ldots \exp Y''_0,$$

then

$$\begin{aligned} X''_k &= X'_k + f(X_0, \ldots, X_{k-1},\, Y_0, \ldots, Y_{k-1}),\\ Y''_k &= Y'_k + g(X_0, \ldots, X_{k-1},\, Y_0, \ldots, Y_{k-1})\end{aligned}$$

with some polynomial functions f and g.

In the case of concern to us, both $\mathfrak{a}$ and $\mathfrak{b}$ are subalgebras of $\mathfrak{n}$. The group corresponding to $\mathfrak{b}$ is $\overline{N}_{\mathbb{Q}_p}(S)$. There is a subgroup $N(S)$ of G such that the group corresponding to $\mathfrak{a}$ is $N_{\mathbb{Q}_p}(S)$. Moreover $N(S)$ is contained in V the unipotent radical of B. As a particular consequence of the lemma

$$N = N_{\mathbb{Q}_p}(S)\overline{N}_{\mathbb{Q}_p}(S).$$

If $\bar{n}_2$ lies in $\overline{N}_{\mathbb{Q}_p}(S)$ then $a_1^{-1}n_1^{-1}\,\bar{n}_2 n_1 a_1$ lies in N. Project it onto $\overline{N}_{\mathbb{Q}_p}(S)$ to obtain $\bar{n}_2'$. It is an easy consequence of the lemma that the map $\bar{n}_2 \to \bar{n}_2'$ is a one-to-one mapping of $\overline{N}_{\mathbb{Q}_p}(S)$ onto itself and that

$$d\bar{n}_2 = \prod_{\substack{\lambda_1(\alpha)<0\\ \alpha>0}} |\xi_\alpha(a_1)|^{-1} d\bar{n}_2'.$$

Now $\psi_\mu(a_1) = p^{\lambda(a_1)(\mu+p)}$ and

$$\rho - \sum_{\substack{\lambda_1(\alpha)<0\\ \alpha>0}} \alpha = \frac{1}{2}\Bigg(\alpha_0 + \sum_{\substack{\lambda_1(\alpha)>0\\ \alpha>0}} \alpha - \sum_{\substack{\lambda_1(\alpha)<0\\ \alpha>0}} \alpha\Bigg) = \frac{1}{2}\Bigg(\alpha_0 + \sum_{\lambda_1(\alpha)>0} \alpha\Bigg).$$

Since

$$\sum_{\lambda_1(\alpha)>0} \lambda(a_1)(\alpha) = 0,$$

we have

$$\int\limits_{\overline{N}_{\mathbb{Q}_p}(R)} \psi_\mu(\bar{n})\,d\bar{n} = \Bigg\{\int\limits_{\overline{N}^0_{\mathbb{Q}_p}} p^{\lambda(a_i)(\mu+\frac{\alpha_0}{2})}\,d\bar{n}_1\Bigg\}\Bigg\{\int\limits_{\overline{N}_{\mathbb{Q}_p}(S)} \psi_\mu(\bar{n}_2)\,d\bar{n}_2\Bigg\}.$$

The first integral is equal to

$$\int\limits_{\mathbb{Z}_p} 1 + \int\limits_{\substack{x\in\mathbb{Q}_p\\ |x|>1}} |x|^{-\mu(H_{\alpha_0})-1}\,dx,$$

which is

$$1 + \left(1-\frac{1}{p}\right)\sum_{n=1}^{\infty} \frac{p^n}{p^{n(\mu(H_{\alpha_0})+1)}} = 1 + \left(\frac{1-\dfrac{1}{p}}{p^{\mu(H_{\alpha_0})}}\right)\frac{1}{1-\dfrac{1}{p^{\mu(H_{\alpha_0})}}}$$

$$= \frac{1-\dfrac{1}{p^{\mu(H_{\alpha_0})+1}}}{1-\dfrac{1}{p^{\mu(H_{\alpha_0})}}}$$

if $\operatorname{Re}\mu(H_{\alpha_0}) > 0$.

5. A REVIEW OF EISENSTEIN SERIES

Let Δ be the set of simple roots of $\mathfrak{h}$. If α_0 belongs to Δ, there is a parabolic group $P = P(\alpha_0)$ of rank one which contains B associated to α_0. Let N be the unipotent radical of P. Then P is the semidirect product of N and a reductive group M. It is convenient to suppose that M contains T. Let A be the center of M and let ${}^0G = A\backslash M$. There is a map from P to ${}^0G$. Furthermore, ${}^0G$ is the adjoint group of a split Lie algebra of rank one less than G. Its Dynkin diagram is obtained by deleting α_0 from the Dynkin diagram of G.

Lemma. *Each of the maps*

$$P_{\mathbb{Q}_p} \to {}^0G_{\mathbb{Q}_p},\, P_{\mathbb{Z}_p} \to {}^0G_{\mathbb{Z}_p},\, P_{\mathbb{A}} \to {}^0G_{\mathbb{A}}$$

is surjective.

It is enough to verify this for the first two maps. Let ${}^0T$ be the image of T in ${}^0G$. Using the Bruhat decomposition one readily shows that the map $P_{\mathbb{Q}_p} \to {}^0G_{\mathbb{Q}_p}$ is surjective if the map $T_{\mathbb{Q}_p} \to {}^0T_{\mathbb{Q}_p}$ is surjective. If ${}^0t$ lies in ${}^0T_{\mathbb{Q}_p}$ then $\xi_\beta({}^0t)$ is given for β in $\Delta - \{\alpha\}$. There is certainly a t in $T_{\mathbb{Q}_p}$ such that $\xi_\beta(t) = \xi_\beta({}^0t)$ for these β; t is mapped to ${}^0t$.

Suppose u in ${}^0G_{\mathbb{Z}_p}$ is the image of $p = vtk$ with v in $V_{\mathbb{Q}_p}$, t in $T_{\mathbb{Q}_p}$, k in $G_{\mathbb{Z}_p}$. For the purposes of the lemma we may suppose that $k = 1$. If α is a root of ${}^0G$, then $|\xi_\alpha(t)| = 1$. Let t_0 be such that $\xi_\alpha(t_0) = |\xi_\alpha(t)|$ for each root α; t_0 must lie in the center of M. Replacing t by t_0t, we may suppose that t lies in $G_{\mathbb{Z}_p}$ or, even better, that t is 1 and $p = v$. If p is infinite, v must lie in $N_{\mathbb{Q}_p}$ and if p is finite v must be congruent modulo $N_{\mathbb{Q}_p}$ to an element of $G_{\mathbb{Z}_p}$. (See C. Chevalley, Séminaire Bourbaki, Exposé 219.)

If p belongs to P let $\chi(p)$ be the determinant of the restriction of Ad p to the Lie algebra of N. Every element of $G_{\mathbb{A}}$ is a product $g = bu$ with b in $B_{\mathbb{A}}$ and u in U; set $\xi_s(g) = \xi_s(b) = \prod_p |\chi(b_p)|^{s+\frac{1}{2}}$. The product is taken over all primes including the infinite one. The function ξ_s is well defined and is a function on $P_{\mathbb{Q}}\backslash G_{\mathbb{A}}$. Let ϕ be one of the basis elements for the cusp forms on ${}^0G_{\mathbb{Q}}\backslash {}^0G_{\mathbb{A}}$. Of course ϕ is supposed to be invariant on the right under ${}^0U$. Also ϕ may be lifted to a function on $N_{\mathbb{A}}P_{\mathbb{Q}}\backslash P_{\mathbb{A}}$. If $g = bu$, set

$$F(g,\, s,\, \phi) = \xi_s(g)\phi(b)\,.$$

This function is well defined. The sum

$$E(g,\, s,\, \phi) = \sum_{\gamma\in P_{\mathbb{Q}}\backslash G_{\mathbb{Q}}} F(\gamma g,\, s,\, \phi)$$

is called an Eisenstein series. It converges absolutely for Re $s > \frac{1}{2}$ but the function on the left is actually a meromorphic function of s for all g. By the way, if g belongs to $G_{\mathbb{R}}$,

$$E(g,\, s,\, \phi) = \sum_{\gamma\in P_{\mathbb{Z}}\backslash G_{\mathbb{Z}}} F(\gamma g,\, s,\, \phi)\,.$$

Suppose $P' = P(\alpha_0')$ is another parabolic group of rank one, and N' is its unipotent radical. Then

$$\int_{N'_{\mathbb{Q}}\backslash N'_{\mathbb{A}}} E(ng,\, s,\, \phi)\, dn$$

is for each g a meromorphic function of s. If Re $s > \frac{1}{2}$, it is equal to

$$\int_{N'_{\mathbb{Q}}\backslash N'_{\mathbb{A}}} \sum_{P_{\mathbb{Q}}\backslash G_{\mathbb{Q}}} F(\gamma ng,\, s\phi)\, dn\,,$$

which equals

$$\int_{N'_{\mathbb{Q}}\backslash N'_{\mathbb{A}}} \sum_{N_{\mathbb{Q}}\backslash G_{\mathbb{Q}}/N'_{\mathbb{Q}}} \sum_{\gamma^{-1}P_{\mathbb{Q}}\gamma\cap N'_{\mathbb{Q}}\backslash N'_{\mathbb{Q}}} F(\gamma\delta ng,\, s,\, \phi)\, dn$$

or

$$\sum_{P_{\mathbb{Q}}\backslash G_{\mathbb{Q}}/N'_{\mathbb{Q}}} \int_{\gamma^{-1}P_{\mathbb{Q}}\gamma\cap N'_{\mathbb{Q}}\backslash N_{\mathbb{A}}} F(\gamma ng,\, s, \phi)\, dn.$$

Because of the Bruhat decomposition we can suppose that each γ is of the form $\gamma = w\gamma'$ with w in the intersection of $G_{\mathbb{Z}}$ and the normalizer of T and γ' in $P'_{\mathbb{Q}_p}$. Then a typical term equals*

$$\int_{w^{-1}\,P_{\mathbb{Q}}w\cap N'_{\mathbb{Q}}\backslash N'_{\mathbb{A}}} F(wng,\, s,\, \phi)\, dn.$$

There is an order on the roots of ${}^0G$ such that the positive roots are those of the form $w\alpha$ where α is a positive root of G. Multiplying w on the left by an element in the normalizer of T in $G_{\mathbb{Z}}\cap M$, we may suppose this is the order induced from the original order on the roots of $\mathfrak{h}$. Let ${}^0\sum_+$ be the roots of ${}^0G$ of the form $w\alpha$ where α is a positive root of G which is not a root of ${}^0G'$ and let ${}^0\sum_0$ be the roots of ${}^0G$ of the form $w\alpha$ where α is a positive root of G which is a root of ${}^0G'$. If α belongs to ${}^0\sum_+$ and β belongs to ${}^0\sum_+$ or to ${}^0\sum_0$ and $\alpha+\beta$ is a root, then $\alpha+\beta$ belongs to ${}^0\sum_+$; on the other hand, if α and β both belong to ${}^0\sum_0$ and $\alpha+\beta$ is a root, then $\alpha+\beta$ belongs to ${}^0\sum_0$. As a consequence, the group N'' whose Lie algebra is the span of $\{X_\alpha|\alpha\in{}^0\sum_+\}$ is the unipotent radical of a parabolic subgroup of ${}^0G$. Since $w^{-1}N''w$ is contained in N' and

$$w^{-1}P_{\mathbb{Q}}w\cap w^{-1}N''_{\mathbb{Q}}w = w^{-1}N''_{\mathbb{Q}}w,$$

our integral equals

$$\int_{(w^{-1}P_{\mathbb{Q}}w\cap N'_{\mathbb{Q}})w^{-1}N''_{\mathbb{A}}w\backslash N'_{\mathbb{A}}} \left\{ \int_{N''_{\mathbb{Q}}\backslash N''_{\mathbb{A}}} F(n_1wng,\, s,\, \phi)dn_1\right\} dn\,.$$

If $wng = bk$ with b in B_A and k in U, the inner integral equals

$$\xi_s(b)\int_{N''_{\mathbb{Q}}\backslash N''_{\mathbb{A}}} \phi(n_1b)\, dn_1$$

which is zero if $N''\neq\{1\}$ because ϕ is a cusp form. Thus the integral vanishes identically unless every positive root of ${}^0G$ is of the form $w\alpha$ where α is a root of ${}^0G'$. Then, if α is a positive root of ${}^0G'$, $w\alpha$ is a linear combination of roots of ${}^0G$ and thus a root of ${}^0G$. As a consequence, $wM'w^{-1} = M$. For these terms we can take $\gamma' = 1$. If $P = P'$, then $w = 1$ is one possibility and the resulting integral is $F(g,\, s,\, \phi)$, which is for each g an entire function of s. The only other possibility is that $wP'w^{-1}$ is the parabolic group opposed to P. This is the case we are interested in. Then $w^{-1}P_Qw\cap N'_Q = \{1\}$ and

$$\int_{N'_{\mathbb{A}}} F(wng,\, s,\, \phi)\, dn$$

is for each g a meromorphic function of s in the whole complex plane.

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* [Added 1970] This statement is, I now notice, an oversimplification. There should be a factor in front which depends on γ and g should be replaced by $\gamma' g$.

We have demanded that w lie in $G_{\mathbb{Z}}$. We can also demand that it lie in $G_{\mathbb{Z}_\infty}$; this will make it easier to evaluate the integral for then w lies in U and

$$F(wng,\, s,\, \phi) = F(wngw^{-1},\, s,\, \phi)\,.$$

It is enough to evaluate the integral for $g = m'$ in $M'_{\mathbb{A}}$. Set $m = wm'w^{-1}$; m lies in $M_{\mathbb{A}}$. A simple change of variable shows that the integral equals

$$\prod\nolimits_p |\chi(m_p)|^{-1} \int\limits_{N'_{\mathbb{A}}} F(mwnw^{-1},\, s,\, \phi)\, dn\,.$$

The product is taken over all primes including the one at infinity. If $\overline{N}$ is the unipotent radical of the group opposed to P, this may also be written as

$$\prod\nolimits_p |\chi(m_p)|^{-1} \int\limits_{\overline{N}_{\mathbb{A}}} F(m\bar{n},\, s,\, \phi)\, d\bar{n}. \tag{7}$$

The map $T \to {}^0T$ determines a map $\mathfrak{h}_{\mathbb{R}} \to {}^0\mathfrak{h}_{\mathbb{R}}$. Since we have more or less consistently viewed ${}^c\mathfrak{h}_{\mathbb{R}}$ and ${}^c({}^0\mathfrak{h})_{\mathbb{R}}$ as the duals of $\mathfrak{h}_{\mathbb{R}}$ and ${}^0\mathfrak{h}_{\mathbb{R}}$, we can agree that this determines a map ${}^c({}^0\mathfrak{h})_{\mathbb{C}}$ into ${}^c\mathfrak{h}_{\mathbb{C}}$. If χ_p is for each p the homomorphism of the Hecke algebra into the complex numbers determined by ϕ, let ${}^0\mu_p$ be one of the elements in ${}^c({}^0\mathfrak{h})_{\mathbb{C}}$ associated to χ_p. Its image in ${}^c\mathfrak{h}_{\mathbb{C}}$ will again be denoted by ${}^0\mu_p$. If ν is the sum of those roots whose root vectors belong to the Lie algebra of N, we set $\mu_p(s) = {}^0\mu_p + s\nu$. Denote this set of roots by $\sum$. If

$$M(s) = \left\{ \prod_{\alpha\in\sum} \frac{\pi^{1/2}\,\Gamma\left(\frac{\mu_\infty(s)(H_\alpha)}{2}\right)}{\Gamma\left(\frac{\mu_\infty(s)(H_\alpha)+1}{2}\right)} \right\} \prod_{p \text{ finite}} \left\{ \prod_{\alpha\in\sum} \frac{1-\dfrac{1}{p^{\mu_p(s)(H_\alpha)+1}}}{1-\dfrac{1}{p^{\mu_p(s)(H_\alpha)}}} \right\},$$

the integral in the expression (7) is equal to

$$M(s)F(m,\, s,\, \phi)\,.$$

This is not too difficult to prove. Observe first that if S_k is the set consisting of the finite prime and the first k finite primes the integral in (7) is equal to

$$\lim_{k\to\infty} \int\limits_{\prod_{p\in S_k}\overline{N}_{\mathbb{Q}_p}} F(m\bar{n},\, s,\, \phi)\, d\bar{n}\,.$$

So to prove our assertion all we need to do is show that, if h lies in $\prod\limits_{q\neq p} G_{\mathbb{Q}_q} \cap G_{\mathbb{A}}$ and m lies in $M_{\mathbb{Q}_p}$ then

$$\int\limits_{\overline{N}_{\mathbb{Q}_p}} F(hm\bar{n},\, s,\, \phi)\, d\bar{n} = \left\{ \prod_{\alpha\in\sum} \frac{\pi^{1/2}\,\Gamma\left(\frac{\mu_p(s)(H_\alpha)}{2}\right)}{\Gamma\left(\frac{\mu_p(s)(H_\alpha)+1}{2}\right)} \right\} F(hm,\, s,\, \phi)$$

if p is the infinite prime and

$$\int\limits_{\overline{N}_{\mathbb{Q}_p}} F(hm\bar{n},\, s,\, \phi)\, d\bar{n} = \left\{ \prod_{\pi\in\sum} \frac{1-\dfrac{1}{p^{\mu_p(s)(H_\alpha)+1}}}{1-\dfrac{1}{p^{\mu_p(s)(H_\alpha)}}} \right\} F(hm,\, s,\, \phi)$$

if p is a finite prime.

Fix a prime, finite or infinite; fix h in

$$\prod_{q\neq p} G_{\mathbb{Q}_q} \cap G_{\mathbb{A}}$$

and consider the function $F(hg,\, s,\, \phi)$, g in $G_{\mathbb{Q}_p}$.

If $h = bu$, b in $B_{\mathbb{A}}$, u in U, if $\bar{b}$ is the projection of b on $M_{\mathbb{A}}$, and if $g = n(g)m(g)k(g)$ with $n(g)$ in $N_{\mathbb{Q}_p}$, $m(g)$ in $M_{\mathbb{Q}_p}$, and $k(g)$ in $G_{\mathbb{Z}_p}$ then it equals

$$\xi_s(h)\xi_s\big(m(g)\big)\phi\big(\bar{b}m(g)\big).$$

If m belongs to $M_{\mathbb{Q}_p}$, set $\psi(m) = \phi(\bar{b}m)$. If it is convenient, we can regard $\psi(m)$ as a function on ${}^0G_{\mathbb{Q}_p}$. We are reduced to evaluating

$$\int\limits_{\overline{N}_{\mathbb{Q}_p}} \xi_s\big(mm(\bar{n})\big)\psi\big(mm(\bar{n})\big)\, d\bar{n} \tag{8}$$

if ψ is a function on ${}^0G_{\mathbb{Q}_p}$ invariant under right translations by elements of ${}^0G_{\mathbb{Z}_p}$ which is an eigenfunction of the operators $\lambda(f)$ for f in ${}^0H_p$ associated to the homomorphism χ_p of ${}^0H_p$ into $\mathbb{C}$ determined by ${}^0\mu_p$. Of course we assume that the integral converges absolutely. Recall that

$$\lambda(f)\psi(g) = \int\limits_{{}^0G_{\mathbb{Q}_p}} \psi(gh)f(h)\, dh$$

if g belongs to ${}^0G_{\mathbb{Q}_p}$.

Let $M = M_{\mathbb{Q}_p}$ and let

$$K = G_{\mathbb{Z}_p} \cap P_{\mathbb{Q}_p}/G_{\mathbb{Z}_p} \cap N_{\mathbb{Q}_p}.$$

Define a measure μ on M/K by setting $\mu(E)$ equal to the measure of

$$\{\bar{n} \in \overline{N}_{\mathbb{Q}_p} \mid m(\bar{n})E\}\,.$$

Suppose k lies in K and $\bar{n}$ in $\overline{N}$ equals $n(\bar{n})m(\bar{n})k(\bar{n})$; let k be the coset of $\bar{k}$. Since

$$\bar{k}\bar{n}\bar{k}^{-1} = \big(\bar{k}n(\bar{n})\bar{k}^{-1}\big)\big(\bar{k}m(\bar{n})\bar{k}^{-1}\big)\big(\bar{k}k(\bar{n})\bar{k}^{-1}\big)\,,$$

the sets $\{\bar{n}|m(\bar{n}) \in kE\}$ and $\{\bar{k}\bar{n}\bar{k}^{-1}|m(\bar{n}) \in E\}$ are the same and μ is left-invariant under K. Define a measure on M, again called μ, which is invariant under left and right translations by elements of K by setting

$$\mu(E) = \int\limits_{M/K} \left\{ \int\limits_K \chi_E(mk)\, dk \right\} d\mu(m)$$

if χ_E is the characteristic function of E.

The integral (8) is equal to

$$\int\limits_M \xi_s(mm_1)\psi(mm_1)\, d\mu(m_1)\,.$$

If F is a subset of ${}^0G_{\mathbb{Q}_p} = A_{\mathbb{Q}_p}\, M_{\mathbb{Q}_p}$ and E is the inverse image of F in $M_{\mathbb{Q}_p}$, set

$$\nu_s(F) = \int\limits_E \xi_s(m_1)\, d\mu(m_1)\,.$$

Since, as we observed earlier, K maps onto ${}^0G_{\mathbb{Z}_p}$, ν_s is invariant on the left and the right under ${}^0G_{\mathbb{Z}_p}$. The integral (8) equals

$$\xi_s(m) \int\limits_{{}^0G_{\mathbb{Q}_p}} \psi(\bar{m}h)\, d\nu_s(h)$$

if $\bar{m}$ is the image of m in ${}^0G_{\mathbb{Q}_p}$.

Let $F_1 \subseteq F_2 \subseteq \ldots$ be an increasing sequence of compact sets (we assume that ${}^0G_{\mathbb{Z}_p} F_i\, {}^0G_{\mathbb{Z}_p} = F_i$) whose union exhausts ${}^0G_{\mathbb{Q}_p}$ and define the measure ν_s^n by $\nu_s^n(F) = \nu_s(F \cap F_n)$. Since ν_s^n belongs to H_p

$$\int\limits_{{}^0G_{\mathbb{Q}_p}} \psi(\bar{m}h)\, d\nu_s(h) = \lim_{n\to\infty} \int\limits_{{}^0G_{\mathbb{Q}_p}} \psi(\bar{m}h)\, d\nu_s^n(h) = \psi(\bar{m}) \lim_{n\to\infty} \chi_p(\nu_s^n)$$

and the integral (8) equals

$$\xi_s(m)\psi(m) \lim_{n\to\infty} \chi_p(\nu_s^n)\,.$$

To evaluate the limit, take ψ to be the function $\psi_{{}^0\mu_p}$ of Section 3; then

$$\xi_s\big(m(g)\big)\psi_{{}^0\mu_p}\big(m(g)\big) = \psi_{\mu_p(s)}(g)$$

and

$$\lim_{n\to\infty} \chi_p(\nu_s^n) = \int\limits_{\overline{N}_{\mathbb{Q}_p}} \psi_{\mu_p(s)}(\bar{n})\, d\bar{n}.$$

The integral on the right can be evaluated by the formula of Gindikin and Karpelevich if $\mathrm{Re}\, s$ is sufficiently large. Retracing our steps, we see that the integral in (7) is indeed equal to $M(s)F(m,\, s,\, \phi)$ and conclude that $M(s)$ is a meromorphic function in the whole complex plane.

J. Tits pointed out a way of expressing $M(s)$ which is more convenient for our purposes. We observed that there was a map of ${}^c({}^0\mathfrak{h})_{\mathbb{C}}$ into ${}^c\mathfrak{h}_{\mathbb{C}}$. It is easy to see that it is induced by an imbedding of ${}^c({}^0\mathfrak{g})$ in ${}^c\mathfrak{g}$. Since ${}^c({}^0G)$ is simply connected, there is an associated map of ${}^c({}^0G)$ into ${}^cG$. Let ${}^c\mathfrak{n}$ be the Lie algebra spanned by the root vectors belonging to positive roots of ${}^cG$ which are not roots of ${}^c({}^0G)$. These are the roots ${}^c\alpha$ corresponding to roots in $\sum$. Let H_{c_α} be the *copoid* attached to c_α and set

$$H_0 = \sum_{\alpha\in\sum} H_{c_\alpha}\,.$$

Let $\mathfrak{n}_1, \ldots, \mathfrak{n}_r$ be the eigenspaces of ad (H_0) in ${}^c\mathfrak{n}$. Let a_i be the eigenvalue of ad (H_0) corresponding to $\mathfrak{n}_i$. Each of the subspaces $\mathfrak{n}_i$ is invariant under ${}^c({}^0G)$; let π_i be the representation of ${}^c({}^0G)$ on $\mathfrak{n}_i$. If $\widetilde{\pi}_i$ is the representation contragredient to π_i, then $M(s)$ can be written as

$$\prod_{i=1}^{r} \frac{\xi(a_i s,\, \widetilde{\pi}_i,\, \phi)}{\xi(a_i s + 1,\, \widetilde{\pi}_i,\, \phi)}\,.$$

6. EXAMPLES

If r is 1, then

$$M(s) = \frac{\xi(a_1 s,\, \widetilde{\pi}_1,\, \phi)}{\xi(a_1 s + 1,\, \widetilde{\pi}_1,\, \phi)}$$

is meromorphic in the whole plane and

$$\xi(s,\, \widetilde{\pi}_1,\, \phi) = M\left(\frac{s}{a_1}\right) \xi(s+1,\, \widetilde{\pi}_1,\, \phi)\,.$$

Since we already know that $\xi(s,\, \widetilde{\pi}_1,\, \phi)$ is analytic in a half-plane, we can conclude that it is meromorphic in the whole plane.

If $r = 2$ and $\xi(s,\, \widetilde{\pi}_1,\, \phi)$ is known to be meromorphic in the whole plane, the same argument shows that $\xi(s,\, \widetilde{\pi}_2,\, \phi)$ is meromorphic in the whole plane. Thus every time we can adjoin a point to the Dynkin diagram of ${}^0G$ to obtain the Dynkin diagram of a group of rank 1 greater, we can expect to find a representation π of ${}^c({}^0G)$ for which $\xi(s,\, \pi,\, \phi)$ is meromorphic in the whole plane. Before listing the possibilities, there is one further remark I should make.

If we define the function ξ'_s in the same manner as ξ_s, the expression (7) is easily seen to equal

$$M(s)\xi'_{-s}(m')\phi(vm'w^{-1}) = M(s)\xi'_{-s}(m')\phi'(m')\,.$$

Recall that $m = vm'w^{-1}$. Of course ϕ' is a function on $A'_{\mathbb{A}}\ M'_{\mathbb{A}}$ and thus a function on ${}^0G'_{\mathbb{A}}$. It satisfies the same conditions as ϕ. Associated to it is a function

$$M'(s) = \prod_{i=1}^{r'} \frac{\xi(a'_i s,\, \widetilde{\pi}_i,\, \phi')}{\xi(a'_i s + 1,\, \widetilde{\pi}'_i,\, \phi')}\,.$$

But $m' \to wm'w^{-1}$ defines an isomorphism of M' with M and an isomorphism of ${}^0G'$ with ${}^0G$. Thus ϕ and ϕ' are essentially the same. Moreover ${}^c({}^0G')$ and ${}^c({}^0G)$ are isomorphic, such that a representation of ${}^c({}^0G')$ may be regarded as a representation of ${}^c({}^0G)$. Recalling that the elements of the adjoint group of ${}^cG$ are orthogonal with respect to the Killing form and that the Killing form turns Ad $w({}^c\mathfrak{n}')$ into the dual of ${}^c\mathfrak{n}$, one sees readily that $r = r'$, that, with a suitable order, $a'_i = a_i$, and that π'_i is the contragredient of π_i. Thus

$$M'(s) = \prod_{i=1}^{r} \frac{\xi(a_i s,\, \pi_i,\, \phi)}{\xi(a_i s + 1,\, \pi_i,\, \phi)}\,.$$

It is known that $M(s)M'(-s) = 1$. This is implied by, but does not imply, the relation $\xi(s,\, \pi_i,\, \phi) = \xi(1-s,\, \widetilde{\pi}_i,\, \phi)$.

In the examples we shall give the Dynkin diagram of G with the points belonging to the Dynkin diagram of ${}^0G$ labeled. We give the number r, the numbers a_i, and the highest weight λ_i of the representations π_i as a linear combination of the fundamental weights δ_j. In the examples considered, π_i is always irreducible. Do not forget that π_i is not a representation of ${}^0G$ but a representation of ${}^c({}^0G)$.

(i)
$$\begin{array}{ccccccccc} 1 & & 1 & & 1 & & 1 & & 1 \\ \circ & — & \circ & — & \circ \ \cdots & & \circ & — & \circ \\ & & \alpha_1 & & \alpha_2 & & \alpha_{n-1} & & \alpha_n \end{array}$$

$r = 1 \qquad a_1 = n+2 \qquad \lambda_1 = \delta_n$

(ii)
$$\begin{array}{ccccccccc} 1 & & 1 & & 1 & & 1 & & 1 \\ \circ & — & \circ \ \cdots & & \circ & — & \circ & — & \circ \\ \alpha_1 & & \alpha_2 & & \alpha_{n-1} & & \alpha_n & & \end{array}$$

$r = 1 \qquad a_1 = n+2 \qquad \lambda_1 = \delta_1$

(iii)

$$\begin{array}{ccccccccccc} 1 & & 1 & & 1 & & 1 & & 1 & & 1 \\ \circ & \text{—} & \circ & \cdots & \circ & \text{—} & \circ & \text{—} & \circ & \cdots & \circ \\ \alpha_1 & & \alpha_2 & & \alpha_{m-1} & & & & \alpha_m & & \alpha_n \end{array}$$

$r = 1 \qquad a_1 = n + 2 \qquad \lambda_1 = \delta_1 + \delta_n$

(iv)

$$\begin{array}{ccccccccc} 2 & & 2 & & 2 & & 2 & & 1 \\ \circ & \text{—} & \circ & \cdots & \circ & \text{—} & \circ & = & \circ \\ \alpha_1 & & \alpha_2 & & \alpha_{n-1} & & \alpha_n & & \end{array}$$

$r = 1 \qquad a_1 = 2(n + 1) \qquad \lambda_1 = 2\delta_1$

(v)

$$\begin{array}{ccccccccc} 1 & & 2 & & 2 & & 2 & & 2 \\ \circ & = & \circ & \text{—} & \circ & \cdots & \circ & \text{—} & \circ \\ & & \alpha_1 & & \alpha_2 & & \alpha_{n-1} & & \alpha_n \end{array}$$

$r = 1 \qquad a_1 = 2(n + 1) \qquad \lambda_1 = 2\delta_n$

(vi)

$$\begin{array}{ccccccccc} 1 & & 1 & & 1 & & 1 & & 2 \\ \circ & \text{—} & \circ & \cdots & \circ & \text{—} & \circ & = & \circ \\ \alpha_1 & & \alpha_2 & & \alpha_{n-1} & & \alpha_n & & \end{array}$$

$r = 2 \qquad a_1 = 2(n + 2) \qquad \lambda_1 = \delta_2$

$\qquad\quad a_2 = n + 2 \qquad \lambda_2 = \delta_1$

(vii)

$$\begin{array}{ccccccccc} 2 & & 1 & & 1 & & 1 & & 1 \\ \circ & = & \circ & \text{—} & \circ & \cdots & \circ & \text{—} & \circ \\ & & \alpha_1 & & \alpha_2 & & \alpha_{n-1} & & \alpha_n \end{array}$$

$r = 2 \qquad a_1 = 2(n + 2) \qquad \lambda_1 = \delta_{n-1}$

$\qquad\quad a_2 = n + 2 \qquad \lambda_2 = \delta_n$

$(viii)$

$$\begin{array}{ccccccccc} & & & & & & & & 1 \\ & & & & & & & & \circ \\ & & & & & & & \diagup & \\ 1 & & 1 & & 1 & & 1 & & \\ \circ & \text{—} & \circ & \cdots & \circ & \text{—} & \circ & & \\ \alpha_1 & & \alpha_2 & & \alpha_{n-2} & & \alpha_{n-1} & & \\ & & & & & & & \diagdown & \\ & & & & & & & & 1 \\ & & & & & & & & \circ \\ & & & & & & & & \alpha_n \end{array}$$

$r = 1 \qquad a_1 = 2n \qquad \lambda_1 = \delta_2$

(ix)

$$\begin{array}{ccccccccc} & & & & & & & & 1 \\ & & & & & & & & \circ \\ & & & & & & & \diagup & \\ 1 & & 1 & & 1 & & 1 & & \\ \circ & \text{—} & \circ & \cdots & \circ & \text{—} & \circ & & \\ \alpha_1 & & \alpha_2 & & \alpha_{n-2} & & \alpha_{n-1} & & \\ & & & & & & & \diagdown & \\ & & & & & & & & 1 \\ & & & & & & & & \circ \\ & & & & & & & & \alpha_n \end{array}$$

$r = 1 \qquad a_1 = 2n \qquad \lambda_1 = \delta_1$

(x)

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & & \circ & & & & \\ 1 & & 1 & & \vert\,1 & & 1 & & 1 \\ \circ & \text{—} & \circ & \text{—} & \circ & \text{—} & \circ & \text{—} & \circ \\ \alpha_1 & & \alpha_2 & & \alpha_3 & & \alpha_4 & & \alpha_5 \end{array}$$

$r = 2 \qquad a_1 = 11 \qquad \lambda_1 = \delta_3$

$\qquad\quad a_2 = 22 \qquad \lambda_2 = 0$

(xi)

$$\begin{array}{ccccccccccc} & & & & & & \overset{1}{\circ} & & & & \\ & & & & & & \vert 1 & & & & \\ \overset{1}{\circ} & \text{——} & \overset{1}{\circ} & \text{——} & \overset{1}{\circ} & \text{——} & \circ & \text{——} & \overset{1}{\circ} & \text{——} & \overset{1}{\circ} \\ \alpha_1 & & \alpha_2 & & \alpha_3 & & \alpha_4 & & \alpha_5 & & \alpha_6 \end{array}$$

$r = 2 \qquad a_1 = 14 \qquad \lambda_1 = \delta_3$

$\qquad\qquad a_2 = 28 \qquad \lambda_2 = \delta_6$

(xii)

$$\begin{array}{ccccccccccc} & & & & \overset{1}{\circ} & & & & & & \\ & & & & \vert 1 & & & & & & \\ \overset{1}{\circ} & \text{——} & \overset{1}{\circ} & \text{——} & \circ & \text{——} & \overset{1}{\circ} & \text{——} & \overset{1}{\circ} & \text{——} & \overset{1}{\circ} \\ \alpha_1 & & \alpha_2 & & \alpha_3 & & \alpha_4 & & \alpha_5 & & \alpha_6 \end{array}$$

$r = 2 \qquad a_1 = 14 \qquad \lambda_1 = \delta_4$

$\qquad\qquad a_2 = 28 \qquad \lambda_2 = \delta_1$

$(xiii)$

$$\begin{array}{ccccccccccccc} & & & & & & & & \overset{1}{\circ} & & & & \\ & & & & & & & & \vert 1 & & & & \\ \overset{1}{\circ} & \text{——} & \overset{1}{\circ} & \text{——} & \overset{1}{\circ} & \text{——} & \overset{1}{\circ} & \text{——} & \circ & \text{——} & \overset{1}{\circ} & \text{——} & \overset{1}{\circ} \\ \alpha_1 & & \alpha_2 & & \alpha_3 & & \alpha_4 & & \alpha_5 & & \alpha_6 & & \alpha_7 \end{array}$$

$r = 3 \qquad a_1 = 51 \qquad \lambda_1 = \delta_1$

$\qquad\qquad a_2 = 34 \qquad \lambda_2 = \delta_6$

$\qquad\qquad a_3 = 17 \qquad \lambda_3 = \delta_3$

(xiv)

$$\begin{array}{ccccccccccccc} & & & & \overset{1}{\circ} & & & & & & & & \\ & & & & \vert & & & & & & & & \\ \overset{1}{\circ} & \text{——} & \overset{1}{\circ} & \text{——} & \circ & \text{——} & \overset{1}{\circ} & \text{——} & \overset{1}{\circ} & \text{——} & \overset{1}{\circ} & \text{——} & \overset{1}{\circ} \\ \alpha_1 & & \alpha_2 & & \alpha_3 & & \alpha_4 & & \alpha_5 & & \alpha_6 & & \alpha_7 \end{array}$$

$r = 3 \qquad a_1 = 51 \qquad \lambda_1 = \delta_1$

$\qquad\qquad a_2 = 34 \qquad \lambda_2 = \delta_2$

$\qquad\qquad a_3 = 17 \qquad \lambda_3 = \delta_5$

(xv)

$$\begin{array}{ccc} \overset{3}{\circ} & \equiv & \overset{1}{\circ} \\ \alpha_1 & & \end{array}$$

$r = 2 \qquad a_1 = 10 \qquad \lambda_1 = 0$

$\qquad\qquad a_2 = 5 \qquad \lambda_2 = 3\delta_1$

This example is particularly striking.

(xvi)

$$\begin{array}{ccc} \overset{3}{\circ} & \equiv & \overset{1}{\circ} \\ & & \alpha_1 \end{array}$$

$r = 3 \qquad a_1 = 3 \qquad \lambda_1 = \delta_1$

$\qquad\qquad a_2 = 6 \qquad \lambda_2 = 0$

$\qquad\qquad a_3 = 9 \qquad \lambda_3 = \delta_1$

$$(xvii)\qquad \begin{array}{ccccccccc} 2 & & 2 & & 2 & & 2 & & 1 \\ \circ & -\!\!- & \circ & \cdots & \circ & \cdots & \circ & =\!\!= & \circ \\ & & \alpha_1 & & \alpha_2 & & \alpha_{n-1} & & \alpha_n \end{array}$$

$r = 2 \qquad a_1 = 2n+1 \qquad \lambda_1 = \delta_1$

$\qquad\qquad a_2 = 4n+2 \qquad \lambda_2 = 0$

$$(xviii)\qquad \begin{array}{ccccccc} 1 & & 1 & & 2 & & 2 \\ \circ & -\!\!- & \circ & =\!\!= & \circ & -\!\!- & \circ \\ & & \alpha_3 & & \alpha_2 & & \alpha_1 \end{array}$$

$r = 2 \qquad a_1 = 22 \qquad \lambda_1 = 0$

$\qquad\qquad a_2 = 11 \qquad \lambda_2 = \delta_3$

$$(xix)\qquad \begin{array}{ccccc} 1 & & 1 & & 2 \\ \circ & -\!\!- & \circ & =\!\!= & \circ \\ & & \alpha_2 & & \alpha_1 \end{array}$$

$r = 1 \qquad a_1 = 4 \qquad \lambda_1 = \delta_2$

$$(xx)\qquad \begin{array}{ccccccccc} 1 & & 1 & & 1 & & 1 & & 2 \\ \circ & -\!\!- & \circ & \cdots & \circ & -\!\!- & \circ & =\!\!= & \circ \\ & & \alpha_1 & & \alpha_{n-2} & & \alpha_{n-1} & & \alpha_n \end{array}$$

$r = 1 \qquad a_1 = 2(n+1) \qquad \lambda_1 = \delta_1$

$$(xxi)\qquad \begin{array}{ccccc} 1 & & 2 & & 2 \\ \circ & =\!\!= & \circ & -\!\!- & \circ \\ \alpha_1 & & \alpha_2 & & \end{array}$$

$r = 2 \qquad a_1 = 5 \qquad \lambda_1 = \delta_2$

$\qquad\qquad a_2 = 10 \qquad \lambda_2 = 0$

$$(xxii)\qquad \begin{array}{ccccccc} 1 & & 1 & & 2 & & 2 \\ \circ & -\!\!- & \circ & =\!\!= & \circ & -\!\!- & \circ \\ \alpha_1 & & \alpha_2 & & \alpha_3 & & \end{array}$$

$r = 2 \qquad a_1 = 16 \qquad \lambda_1 = \delta_1$

$\qquad\qquad a_2 = 8 \qquad \lambda_2 = \delta_3$

$$(xxiii)\qquad \begin{array}{cccccccc} & & & & & & & \overset{1}{\circ}\,\alpha_n \\ 1 & & 1 & & 1 & & 1 & \diagup \\ \circ & -\!\!- & \circ & \cdots & \circ & -\!\!- & \circ & \\ \alpha_1 & & \alpha_2 & & \alpha_{n-2} & & \alpha_{n-2} & \diagdown \\ & & & & & & & \overset{1}{\circ}\,\alpha_n \end{array}$$

$r = 1 \qquad a_1 = 2n \qquad \lambda_1 = \delta_1$

$$(xxiv)\qquad \begin{array}{ccccccccc} & & & & \overset{1}{\circ}\,\alpha_5 & & & & \\ 1 & & 1 & & \vert\, 1 & & 1 & & 1 \\ \circ & -\!\!- & \circ & -\!\!- & \circ & -\!\!- & \circ & -\!\!- & \circ \\ \alpha_1 & & \alpha_2 & & \alpha_3 & & \alpha_4 & & \end{array}$$

$r = 1 \qquad a_1 = 12 \qquad \lambda_1 = \delta_5$

(xxv)

$$\begin{array}{ccccccccc} & & & & \overset{1}{\circ}\,\alpha_4 & & & & \\ & & & & \big|\,1 & & & & \\ \overset{1}{\circ} & \text{——} & \overset{1}{\circ} & \text{——} & \circ & \text{——} & \overset{1}{\circ} & \text{——} & \overset{1}{\circ} \\ \alpha_1 & & \alpha_2 & & \alpha_3 & & \alpha_5 & & \end{array}$$

$r = 1 \qquad a_1 = 12 \qquad \lambda_1 = \delta_4$

$(xxvi)$

$$\begin{array}{ccccccccccc} & & & & & & \overset{1}{\circ}\,\alpha_6 & & & & \\ & & & & & & \big|\,1 & & & & \\ \overset{1}{\circ} & \text{——} & \overset{1}{\circ} & \text{——} & \overset{1}{\circ} & \text{——} & \circ & \text{——} & \overset{1}{\circ} & \text{——} & \overset{1}{\circ} \\ \alpha_1 & & \alpha_2 & & \alpha_3 & & \alpha_4 & & \alpha_5 & & \end{array}$$

$r = 2 \qquad a_1 = 34 \qquad \lambda_1 = 0$

$\qquad\qquad a_2 = 17 \qquad \lambda_2 = \delta_5$

$(xxvii)$

$$\begin{array}{ccccccccccc} & & & & & & \overset{1}{\circ}\,\alpha_5 & & & & \\ & & & & & & \big|\,1 & & & & \\ \overset{1}{\circ} & \text{——} & \overset{1}{\circ} & \text{——} & \overset{1}{\circ} & \text{——} & \circ & \text{——} & \overset{1}{\circ} & \text{——} & \overset{1}{\circ} \\ \alpha_1 & & \alpha_2 & & \alpha_3 & & \alpha_4 & & \alpha_6 & & \end{array}$$

$r = 2 \qquad a_1 = 34 \qquad \lambda_1 = 0$

$\qquad\qquad a_2 = 17 \qquad \lambda_2 = \delta_6$

$(xxviii)$

$$\begin{array}{ccccccccccccc} & & & & & & & & \overset{1}{\circ}\,\alpha_7 & & & & \\ & & & & & & & & \big|\,1 & & & & \\ \overset{1}{\circ} & \text{——} & \overset{1}{\circ} & \text{——} & \overset{1}{\circ} & \text{——} & \overset{1}{\circ} & \text{——} & \circ & \text{——} & \overset{1}{\circ} & \text{——} & \overset{1}{\circ} \\ \alpha_1 & & \alpha_2 & & \alpha_3 & & \alpha_4 & & \alpha_5 & & \alpha_6 & & \end{array}$$

$r = 2 \qquad a_1 = 46 \qquad \lambda_1 = \delta_1$

$\qquad\qquad a_2 = 23 \qquad \lambda_2 = \delta_7$

$(xxix)$

$$\begin{array}{ccccccccccccc} & & & & & & & & \overset{1}{\circ}\,\alpha_6 & & & & \\ & & & & & & & & \big|\,1 & & & & \\ \overset{1}{\circ} & \text{——} & \overset{1}{\circ} & \text{——} & \overset{1}{\circ} & \text{——} & \overset{1}{\circ} & \text{——} & \circ & \text{——} & \overset{1}{\circ} & \text{——} & \overset{1}{\circ} \\ \alpha_1 & & \alpha_2 & & \alpha_3 & & \alpha_4 & & \alpha_5 & & \alpha_7 & & \end{array}$$

$r = 2 \qquad a_1 = 46 \qquad \lambda_1 = \delta_1$

$\qquad\qquad a_2 = 23 \qquad \lambda_2 = \delta_6$

(xxx)

$$\begin{array}{ccccccccccc} & & & & & & \overset{1}{\circ}\,\alpha_6 & & & & \\ & & & & & & \big|\,1 & & & & \\ \overset{1}{\circ} & \text{——} & \overset{1}{\circ} & \text{——} & \overset{1}{\circ} & \text{——} & \circ & \text{——} & \overset{1}{\circ} & \text{——} & \overset{1}{\circ} \\ & & \alpha_1 & & \alpha_2 & & \alpha_3 & & \alpha_4 & & \alpha_5 \end{array}$$

$r = 1 \qquad a_1 = 18 \qquad \lambda_1 = \delta_5$

$(xxxi)$

$$\begin{array}{ccccccccccc} & & & & & & \overset{1}{\circ}\,\alpha_6 & & & & \\ & & & & & & \big|\,1 & & & & \\ \overset{1}{\circ} & \text{——} & \overset{1}{\circ} & \text{——} & \overset{1}{\circ} & \text{——} & \circ & \text{——} & \overset{1}{\circ} & \text{——} & \overset{1}{\circ} \\ & & \alpha_5 & & \alpha_4 & & \alpha_3 & & \alpha_2 & & \alpha_1 \end{array}$$

$r = 1 \qquad a_1 = 18 \qquad \lambda_1 = \delta_1$

$$(xxxii) \qquad \begin{array}{ccccccccccccc} & & & & & & & & \overset{1}{\circ}\,\alpha_7 & & & & \\ & & & & & & & & \big|\,1 & & & & \\ \overset{1}{\circ} & — & \overset{1}{\circ} & — & \overset{1}{\circ} & — & \overset{1}{\circ} & — & \circ & — & \overset{1}{\circ} & — & \overset{1}{\circ} \\ & & \alpha_1 & & \alpha_2 & & \alpha_3 & & \alpha_4 & & \alpha_5 & & \alpha_6 \end{array}$$

$$\begin{array}{lll} r = 2 & a_1 = 29 & \lambda_1 = \delta_1 \\ & a_2 = 58 & \lambda_2 = 0 \end{array}$$