The renormalization fixed point as a mathematical object

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1. Introduction. The success of renormalization group methods in statistical mechanics and, in particular, in the study of critical phenomena is well known to be a consequence of the presence of only a small number of expanding directions, often just one or two, at the pertinent fixed point of the associated infinite-dimensional dynamical system, the other directions being contracting. What cannot be sufficiently emphasized is that the numerical success and the great robustness of the methods appear to result from the extreme rapidity with which the eigenvalues in the contracting directions descend to 0. Given the importance of this property, it is troubling that no methods have been found to establish it rigorously in important concrete cases such as percolation or the Ising model.

It is not immediately clear what is called for, some flexibility certainly. There is of course a dynamical system to define, but its relation to the given model is not prescribed. One might wish, as in the early paper [W] of K. G. Wilson to replace a model on a discrete lattice, for example the Ising model, by a family of Ginzburg-Landau models, more generally a difficult model by an easier model that is more easily imbedded in a family or, for some other reason, more easily treated. Then the appropriate strategy might be to establish the necessary dynamics in this family, enlarged if necessary, and only afterwards transfer the results to the original model, discrete or not. This second step will also be analytical and will presumably rely in turn on a different form of the very characteristics used to establish the properties of the dynamical system.

The fixed point lies in the infinite-dimensional space of the dynamical system and is to be described by coordinates. Since the space is not necessarily linear, these coordinates may not be of the same nature or have the same meaning at the fixed point as they have at the models, which, I recall, are themselves to be regarded as points in the space, but perhaps in a very different part, where the coordinates have quite a different interpretation. The fixed point of the Ising model is, for example, related to a very special conformal field theory, the minimal model with central charge $c = 1/2$. The data defining this field theory must be present in the coordinates of the fixed point, either implicitly or explicitly, but are scarcely to be seen, except by inference, in the model itself.

Another possibility is to search not for an infinite-dimensional system that contains the dynamics of the renormalization but rather for a sequence of finite-dimensional approximations to it. This is largely just a matter of realizing the analytic problems concretely. The second step would then be to transfer the results for this sequence to the original model.

I have thought about these questions over the years, very often in collaboration with Yvan Saint-Aubin and with a number of students at the Université de Montréal, performing some instructive experiments but without making any real mathematical inroads. I would like to take the opportunity of this conference[1] to review the results, and to reflect – in a highly speculative way – on some of the analytic problems that I would like to see solved and on further possible numerical investigations. It is best to turn immediately to the models, for once they are defined, it will be possible to explain in a precise, concrete way what could only be intimated in these introductory remarks.

2. Percolation. There are many models for percolation. In some sense all the usual planar models, for example those discussed in [P2][2], are associated to the same fixed point. Some care has to be taken when interpreting this statement. The group $GL(2,\mathbb{R})$ of linear transformations of the plane operates on the models. So it should operate on the pertinent fixed points of the renormalization dynamics. It turns out, numerically at least, that to each model is associated a conformal structure on the plane and that the appropriate fixed point is determined by this conformal structure. Since the set of conformal structures is a homogeneous space under $GL(2,\mathbb{R})$ that can be identified with the upper half-plane, so is the set of fixed points. To remove this indeterminacy, we consider only models symmetric with respect to both coordinate axes and with respect to interchange of the two axes

To explain the strategy, we fix a model, to be specific, percolation by sites on the square lattice, but any model would do. Recall that, in this model, each site (m,n), $m,n\in\mathbb{Z}$, is open with a probability p, $0\leq p\leq 1$. One interesting value attached to the model is the probability $\pi_L(p)$ that there is a crossing (by leaping from one open site to another open site adjacent in the sense of the lattice) of a large square of side L. There is a unique critical value p_c, $0<p_c<1$, for which

$$0<\liminf_{L\to\infty}\pi_L(p_c)\leq\limsup_{L\to\infty}\pi_L(p_c)<1.$$

For $p<p_c$ both limits are 0; for $p>p_c$ they are both 1. The simplest open problems are whether the limit superior and the limit inferior are equal to each other at $p=p_c$ and whether they are both equal to .5. In comparison to other questions about critical points, they are extremely easy to state, although not necessarily easier to prove. The value .5 is supposed to be universal. It is believed to be valid for

[1] This paper is based on notes for lectures at the Białowieza conference that I was unable, at the last minute, to attend.

[2] The papers [P1, P2, P3] are also available on the website
www.sunsite.ubc.ca/DigitalMathArchive/Langlands/

all other planar models provided they have the three basic symmetries: reflection symmetry in the two axes and symmetry under interchange of the two axes.

There is a more general form of the question ([P2]). Take, once and for all, $p = p_c$. Suppose C is a simple closed curve in the plane and α, β are two intervals on it. If L is any large positive number, we can define as in [P2] the probability $\pi_C^L(\alpha, \beta)$ of a crossing inside the dilation LC of C from $L\alpha$ to $L\beta$. Then it is believed – the belief is supported by the numerical evidence – that the limit, $\pi_C(\alpha, \beta)$ of $\pi_C^L(\alpha, \beta)$ as L approaches infinity exists and is universal. More generally, it should be possible to define a similar limiting probability $\pi_C(\alpha_1, \beta_1, \alpha_2, \beta_2, \ldots; \gamma_1, \delta_1, \gamma_2, \delta_2, \ldots)$ that there are crossings from α_1 to β_1, α_2 to β_2 and so on, but none from γ_1 to δ_1 and so on. Numerical studies give us every reason to believe that these limits, referred to as crossing probabilities, exist and that they are universal, thus independent of the model. This was implicit in [P3], where they, or rather collections of numbers,

$$\rho(E) = \rho_C(\alpha_1, \beta_1, \alpha_2, \beta_2, \ldots; \gamma_1, \delta_1 \ldots),$$

in which E stands for the *event* or *crossing* defined by C and the collection of intervals, are used as coordinates in the space in which the dynamics is defined. In particular, the collection

$$\{\pi(E)\} = \{\pi_C(\alpha_1, \beta_1, \alpha_2, \beta_2, \ldots; \gamma_1, \delta_1, \gamma_2, \delta_2, \ldots)\},$$

in which C runs over all admissible curves and the intervals in C are arbitrary, are supposed to be the coordinates of the pertinent fixed point.

Before coming to [P3], I recall that thanks to Schramm, Smirnov, and several other authors (see [SS] and the papers referred to there) a good deal more is now known than was known when [P2] and [P3] were written. Since the central problem, universality, remains unsolved, it is still possible, none the less, that [P3] has something to offer. It will no doubt be clear to the reader that to overcome the technical difficulties that arise in pursuing the strategy of that paper a much better command of the available techniques than I possess at present will be required. Although life for many of us is not so short as it once was, art too grows longer and ever more rapidly; this has to serve as my apology for presenting my reflections in a half-baked form.

In response to the studies on crossing probabilities reported in [P1], M. Aizenman suggested an hypothesis of conformal invariance for the crossing probabilities. It is still not known that the crossing probabilities are defined for any but a few very special models. It is, for example, not known that they are defined for the square lattice. It is therefore certainly not known that they are universal. Smirnov has, however, proved that the crossing probabilities are defined for percolation on the triangular lattice and that Aizenman's hypothesis of conformal invariance is valid in this case. Thus what remains to be proved is existence in general and universality. Although the papers [P1, P3] were numerical, they were also, for me at least, an attempt to create some confidence in a particular analytic strategy for establishing universality as a consequence of the existence of fixed points for an appropriate renormalizing dynamical system. Although these systems were introduced in [P3], the strategy was not explained. I would like to explain it here, even though I have not yet made any serious attempt to deal with the analytic problems that arise; they are formidable. All I can do is remind myself of them. I begin with a brief review of the definitions of [P3], referring the reader to that paper for more precision.

If conformal invariance is assumed, many of the coordinates $\pi_C(\alpha_1, \dots; \gamma_1, \dots)$ are redundant. In particular, it is enough to take C to be a unit square. For each positive integer l we divide each of its sides into l intervals of length $1/l$. This yields a set $\mathfrak{A}_l$ of $4l$ intervals on the boundary of the square. On the assumption of continuity of the crossing probabilities, it would be enough to know the coordinates $\pi_C(\alpha_1, \beta_1, \alpha_2, \beta_2, \dots; \gamma_1, \delta_1 \dots)$ for intervals α_i, β_i, γ_j and δ_j that are unions of some of the intervals in $\mathfrak{A}_l$, provided of course that l is taken larger and larger. In other words, as an approximation to the full set of crossing probabilities, we can consider only the set defined by the events associated to the unit square and intervals α_i, β_i, γ_j and δ_j each of which is a union of intervals in $\mathfrak{A}_l$.

These events can be defined by a family of basic events. We can attach to each configuration for percolation a function y on pairs (α, β) in $\mathfrak{A}_l$ that takes values in $\{0, 1\}$. The value is 1 if the configuration contains a crossing from α to β and is otherwise 0. The underlying space of the dynamical system is, in principle, the space Π_l of measures on the set $\mathfrak{Z}_l$ of such functions. The insistence in [P3] and [P4] on the FKG-inequality was somewhat of a luxury. It would, however, have been much better to take, as we ultimately did in [P3], only measures that respect the three basic symmetries. So I now add this to the definition of Π_l. It is clear that whenever we can attach crossing probabilities to a given model M of percolation, we can attach to it an element $\eta_l = \eta_l(M)$ of the space Π_l. Moreover, if $l|m$ then we can deduce $\eta_l(M)$ from $\eta_m(M)$ because the intervals in $\mathfrak{A}_l$ are unions of the intervals in $\mathfrak{A}_m$. Finally, universality amounts, at least for crossing probabilities, to the assertion that the family $\{\eta_l \mid l \in \mathbb{N}\}$ is independent of M.

To establish universality it would suffice to show that there is another family ν_l, defined independently of any particular model, such that, for any given model M, the point $\eta_l(M)$ is well approximated by ν_l for large l, a relation more precisely expressed by (7.1) of [P3]. It is not, by the way, supposed that ν_l can be directly deduced from ν_m if $l|m$. This will not be so.

The idea explained, or rather the wish expressed, in [P3] and in [P4] is that ν_l could be introduced as the fixed point of a transformation defined independently of any model. The transformation, which is intended to be a finite-dimensional approximation to the dynamics of renormalization, was defined in [P3] and a fixed point was exhibited numerically for $l = 2$, but no attempt was made to begin the analysis.

There is no need to consider all $l > 0$. It clearly suffices, for the purpose of establishing universality, to consider any sequence $\{l_k\}$ of integers that approaches infinity multiplicatively, for example, the sequence 2^k, $k \geq 0$. It was also necessary to replace $\mathfrak{Z}_l$ by a subset $\mathfrak{Y}_l$, or rather to demand that all the probability measures in Π_l assign the measure 0 to points outside $\mathfrak{Y}_l$. This is easy to arrange even for the measures associated to percolation models, but begs a question that, sooner or later, will come back to haunt anyone who attempts to apply the strategy. In essence, the observation is that the finite model cannot be an approximation to percolation if connections between neighboring intervals are admitted indiscrimately. So we considered only the functions in a set $\mathfrak{Y}_l$ that is defined by excluding most such connections. This too entails possible difficulties and it has still to be shown that they do not arise.

To explain this question, consider the dynamical transformation $\Theta_l = \Theta_l^{(2)} : \Pi_l \to \Pi_l$ introduced in [P3, P4]. A basic object is the square with its boundary divided into $4l$ intervals of equal size. Suppose we fit four such squares together to form a single large square. The intervals in the sides of the smaller squares that lie on the boundary of the large square will divide it into $8l$ intervals of equal size. We fuse adjacent intervals in pairs to arrive at a division of the boundary of the large square into $4l$ equal intervals.

If we have for each of the small squares $\sigma_{i,j}$ a point $y_{i,j}$ in $\mathfrak{Y}_l$, then we can try, using just the crossings of $y_{i,j}$, to cross from one of the $4l$ intervals on the boundary of the large square to another, the understanding being that we connect a crossing of $\sigma_{i,j}$ to one of $\sigma_{i',j'}$ if these two small squares have a common side and if the two points $y_{i,j}$ and $y_{i',j'}$ both reach a common interval in the common side. In this way, we attach to the collection $\{y_{i,j}\}$ an element of $\mathfrak{Z}_l$. Modifying it by removing the connections between adjacent sides, we arrive finally at a point in $\mathfrak{Y}_l$. This map of the 4-fold product of $\mathfrak{Y}_l$ with itself to $\mathfrak{Y}_l$ yields immediately the associated map Θ_l on measures and it is this map that defines the dynamics and for which we need to establish the existence of a fixed point ν_l.

We need to establish not only the existence of a sequence of fixed points

$$\nu_{l_1}, \nu_{l_2}, \ldots$$

as $l_1, l_2, \ldots$ runs through a sequence of positive integers tending multiplicatively to infinity, but also that, for any given model M and for each l in the sequence, ν_l approaches $\eta_l = \eta_l(M)$, or rather that, in the notation of [P3],

$$\lim_{k\to\infty} \Gamma_m^{l_k}(\nu_{l_k}) = \eta_m.$$

The map $\Gamma_m^{l_k}$ is the map on measures attached to the coarsening map $\mathfrak{A}_{l_k} \to \mathfrak{A}_m$, thus to the coarsening map $\mathfrak{Z}_{l_k} \to \mathfrak{Z}_m$. A small, but essential, technical point aside, this is defined by the condition that the image of a function z joins two intervals in $\mathfrak{A}_m$ precisely when these two intervals contain intervals in $\mathfrak{A}_{l_k}$ joined by z.

We have therefore to show that, among other things, η_l is an approximate fixed point of the transformation Θ_l. Since we pass from $\mathfrak{Z}_l$ to $\mathfrak{Y}_l$ by suppressing connections between adjacent intervals, we will have to establish quantitative forms of the following type of assertion. Take the unit square, divide it in two parts by a central vertical line, and divide this vertical line into l equal intervals. Then, for l large, the probability that there is a horizontal crossing of the square is approximately the probability that there is a horizontal crossing without any subpath that moves from one of the l equal subintervals to an adjacent one.[3]

The existence of a sequence of fixed points will perhaps be most easily established for the indices $l_k = 2^k$. Because of the numerical results of [P3], it can, in some sense, be taken for granted that ν_2 exists and even, although this is not of much use, that it is close to η_2. Since we are now taking only measures fixed by the three basic symmetries, many of the eigenvalues appearing in Table III of [P3] are no longer pertinent. The first pertinent ones are 1.6345851, 0.4072630, 0.2445117, 0.1721207, 0.1123677. Thus there is only one that is greater than 1. It is the only eigenvalue that is relevant in the technical sense. The others are less than 1 and, apparently, rapidly decreasing to 0. The proposed strategy is to begin with the fixed point at level $k = 1$ and to establish the existence of the fixed point for larger k inductively using Newton's method. This will require, of course, showing that the eigenvalue structure, one relevant eigenvalue and a rapidly decreasing sequence of irrelevant eigenvalues, is preserved. Although I briefly outline this strategy in the following paragraphs, I stress immediately, once and for all, that I have not, perhaps to my shame, begun to think about the estimates that will be needed. They will not be easy to establish.

Given any point y_{2^k} in $\mathfrak{Y}_{2^k}$, we can construct, by the heaping $\Phi_{2^k}^{(2)}$ of [P3], a point in $\mathfrak{Y}_{2^{k+1}}$, basically by the same process as before, except that we map the 4-fold product of $\mathfrak{Y}_{2^k}$ not to $\mathfrak{Y}_{2^k}$ but

[3] There are mathematicians, for example O. Schramm or S. Smirnov, who have thought much more deeply about such questions than I and who have, I believe, partial answers to them.

to $\mathfrak{Y}_{2^{k+1}}$, as is possible because the side of each square $\sigma_{i,j}$, $i,j=1,2$, that lies in the boundary of the large square is already divided into 2^k sides. There is also, as in [P3], a coarsening $\Gamma_{2^k}^{2^{k+1}}$ that maps $\mathfrak{Y}_{2^{k+1}}$ to $\mathfrak{Y}_{2^k}$. The associated maps on measures yield a repeating sequence

$$\rightarrow \Pi_{2^k} \rightarrow \Pi_{2^{k+1}} \rightarrow \Pi_{2^k} \rightarrow \Pi_{2^{k+1}} \rightarrow .$$

Let $\Psi_k : \Pi_{2^{k+1}} \rightarrow \Pi_{2^{k+1}}$ be the composition $\Phi_{2^k}^{(2)} \circ \Gamma_{2^k}^{2^{k+1}}$ of the two distinct maps in this sequence and, to simplify notation, let $\Delta_k = \Theta_{2^{k+1}}$. It is $\Gamma_{2^k}^{2^{k+1}} \circ \Phi_{2^k}^{(2)}$. Suppose a fixed point ν_{2^k} of Θ_{2^k} has been found. Let ψ_k be its image in $\Pi_{2^{k+1}}$. It is clear that ψ_k is a fixed point of Ψ_k.

The map Ψ_k is a projection $\Gamma_{2^k}^{2^{k+1}}$ on Π_{2^k} followed by the map $\Phi_{2^k}^{(2)}$. The tangent map $D\Gamma_{2^k}^{2^{k+1}}$ at ψ_k and, indeed, at any point is also a projection. So the behavior of $D\Psi_k$ is, apart from a preliminary compression, that of $D\Phi_{2^k}^{(2)}$. Thus, in order to show that $D\Psi_k$ has, apart from a large number of additional very small eigenvalues, eigenvalues close to those of $D\Theta_{2^k}$, thereby beginning the induction, we have to show that $D\Gamma_{2^k}^{2^{k+1}}$ does not deform the image of Π_{2^k} in $\Pi_{2^{k+1}}$, at least not in a neighborhood of the fixed point ν_{2^k}.

We can express this in terms of matrices. Suppose we choose coordinates in Π_{2^k} and $\Pi_{2^{k+1}}$ so that

$$D\Gamma_{2^k}^{2^{k+1}} = \begin{pmatrix} I & 0 \end{pmatrix}.$$

Let

$$\begin{pmatrix} A \\ B \end{pmatrix}$$

be the matrix of $D\Phi_{2^k}^{(2)}$ at ν_{2^k}. Then

$$D\Psi_k = \begin{pmatrix} A & 0 \\ B & 0 \end{pmatrix}, \qquad D\Theta_{2^k} = A.$$

So we seem to need to show that B is of the form CA, for then

$$\begin{pmatrix} I & 0 \\ C & I \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} I & 0 \\ -C & 0 \end{pmatrix} = \begin{pmatrix} A & 0 \\ B & 0 \end{pmatrix}.$$

It will be important to control the size of C.

So long as we have not fixed the metrics on Π_{2^k} and $\Pi_{2^{k+1}}$, this is not a meaningful demand. What it might mean ultimately is that if α and β are intervals of length $1/2^k$ on the boundary and α_1, α_2, respectively β_1, β_2, the two subintervals of length $1/2^{k+1}$ into which they can be divided and if π' is a measure near the fixed point, π its image under $\Phi_{2^k}^{(2)}$, and π'' the image of π under $\Gamma_{2^k}^{2^{k+1}}$, then the probability,with reference to π, that α_i is connected to β_j is approximately independent of i and j and approximately determined, in a universal way, thus independently of α and β, by the probability that α is connected to β.

Our goal is not to find a fixed point of Ψ_k but of Δ_k. The method of Newton, in which one establishes that a map

$$\pi \to \pi - D^{-1}(\Delta_k \pi - \pi),$$

D being a constant approximation to the tangent map $D\Delta_k$, is contracting on some domain, is the obvious technique to apply ([L]). To make it work here, we would like to show that Δ_k is close to Ψ_k.

The maps Δ_k and Ψ_k on measures are both attached to maps

$$\mathfrak{Y}_{2^{k+1}} \times \mathfrak{Y}_{2^{k+1}} \times \mathfrak{Y}_{2^{k+1}} \times \mathfrak{Y}_{2^{k+1}} \to \mathfrak{Y}_{2^{k+1}}$$

For the first, we heap and then coarsen; for the second, we coarsen and then heap. Were it not that some connections are suppressed upon coarsening, the second would always yield an element of $\mathfrak{Y}_{2^{k+1}}$ connecting more pairs of intervals. There would not, in general, be many more connections if we could be certain of a condition that I now attempt to explain.

Suppose we have two abutting unit squares T_1, T_2 and in their common side one of the intervals α of length $1/2^k$ into which it is divided. Let α_1 and α_2 be the two halves of α, both of length $1/2^{k+1}$. Suppose y_1 and y_2 lie in $\mathfrak{Y}_{2^{k+1}}$, the first with respect to one of the two squares and the second with respect to the other and suppose y_1 connects some interval β_1 to α_1 and y_2 connects some interval β_2 to α_2. Then we want there to be some other interval γ of length $1/2^{k+1}$ on the common side such that, for $i = 1$ and $i = 2$, y_i connects β_i to γ. We cannot expect this always to be so, but we would like it be so for most β_i and most β_j with a probability in y_1 and y_2 that is almost 1 with respect to the measure ψ_k, and thus with respect to any measure close to ψ_k.

Once again, no metric has been defined, but any metric on the measures will have to regard two measures as close not only if they assign approximately the same measure to each set but also if they assign equal measures to approximately the same set. Thus two atomic measures $c_1\delta_{y_1}$ and $c_2\delta_{y_2}$ will have to be regarded as close if c_1 is close to c_2 and y_1 is close to y_2. So this condition appears to be what is necessary to show that when acting upon a neighborhood of ψ_k, Δ_k is close to Ψ_k.

The measure ψ_k is the image of ν_{2^k} and ν_{2^k} is supposed to be an approximation to η_{2^k}. Thus ψ_k can be expected to be an approximation to $\eta_{2^{k+1}}$. The required property with respect to this measure is, at least roughly and intuitively, a consequence of a well known property of critical percolation on, say, the triangular lattice, which it will be necessary to establish anew for ν_{2^k} or ψ_k in the context of finite-dimensional approximations to percolation. Suppose each y_i is defined by an occupied percolation path p_i. Consider the square S_1 of side $1/2^{k+1}$ whose center is the common endpoint of α_1 and α_2. Let S_2 be a second square with the same center and a side $a/2^{k+1}$, where a is chosen as large as possible with respect to the condition that neither β_1 nor β_2 meets S_2. Then, as a result of Lemma 7.2 of [K], with a very high probability (of the order of $1 - a^{-\delta}, \delta > 0$) the square S_2 contains a path surrounding S_1. This path together with the union of the connected path in p_1 from β_1 to α_1 and the connected path in p_2 from β_2 to α_2 would, for the triangular lattice, be an occupied path joining β_1 to β_2.

Whether this strategy or something completely different will ultimately be used to establish universality in percolation, I do not know. Nor do I know whether I shall ever return to the problem in a serious way. It is nevertheless a pleasure to remind myself now and again of its depth.

3. The Ising model. The paper [I] is rather long and some of the central numerical conclusions are easy to overlook. There is one in particular that I want to recall here. Since we are passing to a different topic, all notation is again free.

Various forms of the Ising model were considered in [I]. They are all defined by a graph Γ on a surface S, closed or open, with or without boundary. If Π is the set of vertices of Γ, then each model assigns a probability to each configuration $\sigma : \Pi \to \pm 1$, thus to each configuration of spins. Each configuration defines a unique partition of the vertices into the maximal subsets of constant sign that are connected within Γ. For the models of [I], it was possible to attach to each such partition a collection of simple oriented curves, the contour lines, $L_1, L_2, \ldots$. There will be, in general, several such collections attached to each σ because the orientations are arbitrary. Moreover, for most models there are configurations for which even the unoriented curves are ambiguously defined. Thus to each σ is attached the finite set Λ_σ of such collections and to each element $\lambda = \{L_1, L_2, \ldots\}$ of Λ_σ a probability. Set $\Lambda = \cup_\sigma \Lambda_\sigma$. It is a set furnished with a probability.

There was, in addition, for each model a notion of mesh ϵ and the possibility existed of taking the mesh to 0. Suppose we have on S an oriented curve C that is, at least at first, closed and smooth (even implicitly analytic) although not necessarily connected. Thus it is the union of a finite number of simple closed curves. Although some care has to be taken with perhaps degenerate intersections, it is pretty clear how to attach to each collection $\lambda = \{L_1, L_2, \ldots\}$ a distribution δ_λ on C. The distribution will, in fact, be a measure, the sum of atomic measures of mass ± 1 at each of the intersections of each L_i with C, the sign being determined by the relative orientation of L_i and C at the given intersection.

The map $\lambda \to \delta_\lambda$ allows us to transfer the probability measure on Λ to a probability measure on the space of distributions on C. This measure we denote μ_ϵ to emphasize the dependence on ϵ. What some of the numerical experiments of [I] demonstrate, or at least were meant to demonstrate, is that

$$\lim_{\epsilon \to 0} \mu_\epsilon = \mu = \mu_C = \mu_C^S$$

exists as a measure on the space of distributions on C and that it is conformally invariant and universal. As in percolation, the conformal invariance is with respect to a conformal structure determined by the model. Universality is, of course, also valid only within the family of models defining a given structure. It is important to observe that the measure will depend on S. It will be important to determine the extent of this dependence.

The experiments establishing the conformal invariance were not nearly so extensive as those undertaken in [P2] for percolation. Moreover, conformal invariance refers now to a structure with more components: the surface S, which may have a boundary, and the curve C, which may or may not lie in that boundary. So the conformal invariance is sometimes of a different nature than the conformal invariance for percolation: the pertinent maps refer to a surface and a curve, not simply to a curve and its interior. For the present purposes, the most important case is that of a compact S without boundary, which was experimentally the most difficult case of [I] and also the one to which the least space was given. Indeed, in that paper it is little more than an afterthought.

What I want to do here is to take the existence of μ_C for granted and suggest further properties that it might possess and that might be tested or even established, although proving that it has these properties is likely to be much harder than the problems for percolation discussed in the previous section. The measure μ_C depends strongly on the way C lies in S. We shall here be concerned primarily with compact S without boundary, thus, for example, with the plane compactified to the Riemann sphere or with an infinitely long cylinder also compactified to the sphere.[4] Numerical experiments for these examples were discussed toward the end of §3.2 of [I]; they are for me the most suggestive of the paper. The compact surface S is implicitly endowed with a conformal structure that is determined by the model – or universality class of models – with which we begin.

When considering percolation, we supposed implicitly that we were dealing with translation-invariant models in the plane. So the resulting conformal structures were parametrized by the upper

[4] As a consequence, the measures μ_ϵ and the limiting measure μ_C are concentrated on distributions that annihilate the constant functions. There are other possibilities that lead, in the language of conformal field theory, to different sectors. They will be ignored.

half-plane and it was natural to separate the models into universality classes according to the attached conformal structure. When allowing, as now for the Ising model, models on various surface, each model defining on the surface one of the very many possible conformal structures on it, the division into universality classes is not so natural. It is perhaps somewhat better to treat all models as belonging to one universality class and to consider whatever data, parameters, or set of coordinates that define the fixed point as referring to all possible S and all possible conformal structures on them. As an example, whatever information we have to give to define the fixed point must allow us to generate all the measures μ_C. For translation-invariant percolation the matter is at first glance simpler as the symmetric fixed point generates under the action of $GL(2,\mathbb{R})$ all the others. More general models are, however, attached to less simple conformal structures ([P2]). So, at least in principle, to determine fully the universal fixed-point even for percolation requires that we be able to calculate crossing probabilities not only in the plane but also on surfaces with other conformal structures.

For percolation and for the Ising model, there is abundant evidence that there are also conformal field theories attached to the universality class. What I want to discuss in the remainder of this paper is their possible relation to the measures μ_C^S, confining myself to the Ising model for which the evidence of [I] is available. There would be several steps to the construction. First of all, a Hilbert space $\mathcal{H}$ has to be attached to the parametrized boundary of the unit disk. It will be defined as an L^2-space with respect to one of the measures μ_C^S. The space $\mathcal{H}$ introduced, there is a second, more recondite, collection of objects to be defined. If the compact surface S with conformal structure together with two families of parametrized, smooth, oriented, simple closed curves $\{C_1,\dots,C_m\}$ and $\{C_1',\dots,C_n'\}$ on it, each of these $m+n$ curves disjoint from all the others and if $\Sigma\subset S$ has as oriented boundary the first set of curves as oriented together with the other set of curves oppositely oriented, then there is an operator

$$K_\Sigma : \otimes_{i=1}^m\mathcal{H}\to\otimes_{j=1}^n\mathcal{H}$$

attached to Σ. These operators are, in fact to depend only on Σ and not on the closed surface S.

They are also to be multiplicative, in the sense that if Σ' is a second surface with boundary $\{C_1',\dots,C_n'\}$, with the opposite orientation, and $\{C_1'',\dots,C_p''\}$, then

$$K_{\Sigma'}\circ K_\Sigma=\alpha K_{\Sigma''}, \tag{1}$$

if Σ'' is obtained by pasting Σ and Σ' along $\cup_{j=1}^n C_j'$. The relation (1) is a projective relation, valid for some constant α.

There is to be in addition an action of the circle group $e^{i\theta} \to \pi(e^{i\theta})$ on $\mathcal{H}$. Taking as Σ the annulus with inner radius 1 and outer radius e^r and setting $\pi(e^r) = K_\Sigma$, we see from (1) that $\pi(e^r)\pi(e^t) = \pi(e^{r+t})$. These two actions together will yield a representation of the semigroup $\{z \in \mathbb{C} \,|\, |z| > 1\}$. More generally, as in the papers of G. Segal ([S]), the operators K_Σ will yield an action of the semigroup of annuli with parametrized boundaries and then an action of the direct sum of two copies of the Virasoro algebra, one holomorphic, one antiholomorphic. There is a final step, the factorization of this representation into a direct sum of tensor products of an irreducible representation of the holomorphic algebra and one of the antiholomorphic algebra, and then the factorization of K_T in general into the contribution of a holomorphic conformal field theory and the contribution of an antiholomorphic field theory. This last step is very elaborate even for such a simple model as the free boson (see [CG] and the papers there referred to); there is no point at this stage, at least not for me, in speculating on it for the Ising model. I do, however, find it essential to be quite clear about those properties of the measures μ_C^S that permit the introduction of K_Σ. So I begin by reviewing them in the context of the free boson. They are simple enough theoretically and undoubtedly commonplaces for specialists, but I have no suitable reference. What is perhaps perfectly obvious to others was not always so to me.

Suppose $C = \cup_{i=1}^n C_i$ is the union of disjoint simple curves and is contained in the compact surface S. We take S to be without boundary at first, although a similar construction can be made when it has a boundary, even when some of the curves C_i lie in its boundary. If φ is a smooth function on C then we extend it to a function φ_S that is harmonic on each component of the complement of C in S with boundary values φ. The Dirichlet form

$$D(\varphi) = D(\varphi_S) = \frac{1}{2\pi}\int_S \{ \big(\frac{\partial \varphi_S}{\partial x}\big)^2 + \big(\frac{\partial \varphi_S}{\partial y}\big)^2 \} dxdy$$

is a quadratic form that depends only on the conformal structure. If g is any positive constant, we can introduce the gaussian measure on the distributions that annihilate the constant functions that is defined by

$$\exp(-gD(\varphi)).$$

This is the measure μ_C^S attached to C in S for the free boson. A property of μ_C^S that seems to be very important is that its equivalence class, in the sense of mutual absolute continuity, is independent of S, provided that S is compact without boundary.

In essence, this means that if S_1 and S_2 are two such surfaces, then the difference $D(\varphi_{S_1}) - D(\varphi_{S_2})$, defined at first only for sufficiently smooth φ, can be extended to all distributions, or at least, given our restrictions, to all distributions annihilating the constant functions, or even, the weakest possible assertion, just to all distributions outside a set of measure 0 with respect to the gaussian measure. Before sketching a proof of this, consider some examples.[5]

Consider first the unit circle C in the Riemann sphere. Let

$$\varphi(z) = \sum_{k>0} a_k z^k + \sum_{k>0} a_{-k} \bar{z}^k, \quad |z| = 1. \tag{2}$$

The function φ extends as φ_{S_l} to $S_l = \{|z| \leq 1\}$ in the form given. It extends to $S_r = \{|z| \geq 1\}$ as

$$\varphi_{S_r} = \sum_{k>0} a_k \bar{z}^{-k} + \sum_{k>0} a_{-k} z^{-k}.$$

The form

$$D(\varphi_S) = D(\varphi_{S_l}) + D(\varphi_{S_r}). \tag{3}$$

The two terms are easily calculated as, for example, in [E]. The first and second terms on the right are both equal to

$$\sum_{k=1}^{\infty} 2k a_k a_{-k}. \tag{4}$$

So (3) is

$$\sum_{k=1}^{\infty} 4k a_k a_{-k}. \tag{5}$$

We can also imbed the unit circle in a torus S_A by taking the annulus S bounded by $C_1 = \{z \,|\, |z| = 1\}$ and by $C_2 = \{z \,|\, |z| = A\}$, $A > 1$, and by identifying $e^{i\theta}$ with $Ae^{i\theta}$. Then the function φ will be defined on C_1 by (2) and on C_2 by

$$\varphi(z) = \sum_{k>0} \frac{a_k}{A^k} z^k + \sum_{k>0} \frac{a_{-k}}{A^k} \bar{z}^k, \quad |z| = A. \tag{6}$$

We apply (6.4) of [E] with $q = A^{-1}$ to obtain

$$D(\varphi_S) = \sum_{k=1}^{\infty} 4k a_k a_{-k} - \sum_{k=1}^{\infty} 8 a_k a_{-k} \frac{q^k}{1 - q^{2k}}. \tag{7}$$

[5] In the calculations, I have suppressed the constant term of the functions φ. This makes the formulas more transparent and suffices for our purposes. The formula can easily be extended to include the constant terms ([E]).

The two results are not the same, but the difference is a series that converges for the Fourier coefficients of any distribution on the circle. As a result the measure $\mu_C = \mu_C^S$ obtained from the Riemann sphere S is absolutely continuous with respect to any of the measures $\mu_C^{S_A}$ and conversely.

We make a similar calculation for the annulus T bounded by the two circles $C_i = \{z| \, |z| = A_i\}$, $A_1 < A_2$. We first of all imbed it in the Riemann sphere. The function φ on $C = C_1 \cup C_2$ is given by specifying it as φ_i on C_i. I suppose, for simplicity, that the constant terms of each φ_i are 0. Let $\{a_k/A_1^{|k|}\}$ be the Fourier coefficients of φ_1 and $\{b_k/A_2^{|k|}\}$ those of φ_2. The Riemann sphere is the union of the three regions, $S_l = \{z| \, |z| < A_1\}$, T, $S_r = \{z| \, |z| > A_2\}$. It is clear that

$$D(\varphi_S) = D(\varphi_{S_l}) + D(\varphi_T) + D(\varphi_{S_r}).$$

The first and third terms on the right are calculated from (4), the second from (6.4) of [E]. If $q = A_1/A_2$, the result, upon simplifying, is that $D(\varphi_S)$ equals

$$\sum_{k=1}^{\infty} 2k\{a_k a_{-k}\frac{2}{1-q^{2k}} - a_k b_{-k}\frac{2q^k}{1-q^{2k}} - b_k a_{-k}\frac{2q^k}{1-q^{2k}} + b_k b_{-k}\frac{2}{1-q^{2k}}\}.$$

Apart from a term that converges for the Fourier coefficients of any distribution on C this is the result that would be obtained if we took the disjoint union $\tilde{S}$ of two Riemann spheres and imbedded C_1 in the first and C_2 in the second to obtain an imbedding of C in $\tilde{S}$. So μ_C^S and $\mu_C^{\tilde{S}}$ are mutually absolutely continuous.

We can calculate μ_C with respect to yet another surface by taking $A = A_3 > A_2$ and identifying $A_1 e^{i\theta}$ with $A_3 e^{i\theta}$ to form a torus S_A. We again calculate $D(\varphi_{S_A})$ with the help of (6.4) of [E]. If $q_1 = A_1/A_2$ and $q_2 = A_2/A_3$, the result is the sum of

$$\sum_{k=1}^{\infty} 4k\{a_k a_{-k} + b_k b_{-k}\}$$

and

$$-\sum_{i=1}^{2}\sum_{k=1}^{\infty} 4k\{a_k a_{-k}\frac{q_i^{2k}}{1-q_i^{2k}} + a_k b_{-k}\frac{2q_i^k}{1-q_i^{2k}} + b_k a_{-k}\frac{2q_i^k}{1-q_i^{2k}} + b_k b_{-k}\frac{q_i^{2k}}{1-q_i^{2k}}\}.$$

Once again, this second term converges for the Fourier coefficients of any distribution.

It is, more generally, easy to see that if $C = \cup_{j=1}^n C_j$ is the disjoint union of n simple closed smooth curves imbedded in a compact Riemann surface S without boundary, then the absolute continuity class of the measure μ_C^S is independent of S. I present a rough argument. Suppose that S_1 and S_2 are two such surfaces to which we give metrics compatible with the Riemannian structure. I show only that $D(\varphi_{S_1}) - D(\varphi_{S_2})$ is defined for any distribution φ. We parametrize C, or more precisely we parametrize the curves C_i. Suppose that $T = \cup_{j=1}^n T_j$ is a disjoint union of open annuli such that the imbedding of C in S_i, $i = 1, 2$ extends to a holomorphic imbedding of T into the same S_i. Suppose ϵ is a smooth function on T that is 1 in a neighborhood of C and 0 near the boundary of T. If φ is given on C, let φ_{S_i} be the harmonic function on $S_i - C$ with boundary values φ. The function $\epsilon\varphi_{S_i}$ can be regarded as a function on T or on S_1 and can also be transported to S_2. If Δ_2 is the Laplacian on S_2, then

$$\Delta_2(\varphi_{S_2} - \epsilon\varphi_{S_1}) = -\Delta_2(\epsilon\varphi_{S_1}). \tag{8}$$

Take, at first, S_1 to be a disjoint union of Riemann spheres. If φ is given by (2) on the curve C_i, then it is clear that, even if φ is a distribution, the function φ_{S_1} is defined outside of T and that we can bound any derivative of φ_{S_1} in the region in $\cup T_j$ where $\epsilon \neq 1$, the bound depending of course on the distribution φ. If φ is given on C_j by (2) with coefficients a_k^j, then, for an appropriate integer $m > 0$, it can be taken to be a universal constant A_m times

$$\sum_{j=1}^{n}\sum_{k=1}^{\infty}(|a_k^j| + |a_{-k}^j|)k^{-m}.$$

Moreover, the right side of (8) is 0 where $\epsilon = 1$. As a consequence, we can bound the derivatives of $\Delta_2(\epsilon\varphi_{S_1})$, which is of course 0 where $\epsilon = 0$. By the standard theory of elliptic differential equations, we can majorize the function $\varphi_{S_2} - \epsilon\varphi_{S_1}$, which vanishes on T, and its derivatives, and thus $D_2(\varphi_{S_2} - \epsilon\varphi_{S_1})$, in terms of the original distribution φ. We can also, of course, majorize the derivatives of $\varphi_{S_1} - \epsilon\varphi_{S_1}$. To majorize $D_1(\varphi_{S_1}) - D_2(\varphi_{S_2})$, we observe that it is equal to the sum of three terms: the difference

$$D_1(\varphi_{S_1} - \epsilon\varphi_{S_1}) - D_2(\varphi_{S_2} - \epsilon\varphi_{S_1}) \tag{9}$$

and the difference of the two cross-terms coming from polarization,

$$\frac{1}{2\pi}\int \frac{\partial}{\partial x}(\varphi_{S_j} - \epsilon\varphi_{S_1})\frac{\partial}{\partial x}(\epsilon\varphi_{S_1}) + \frac{\partial}{\partial y}(\varphi_{S_j} - \epsilon\varphi_{S_1})\frac{\partial}{\partial y}(\epsilon\varphi_{S_1}). \tag{10}$$

There is now no difficulty in bounding (9) and to bound (10) all we need to do is to integrate sufficiently often by parts, inserting an appropriate partition of unity. Thus $\mu_C^{S_1}$ and $\mu_C^{S_2}$ are mutually absolutely continuous when S_1 is chosen in the way indicated. By transitivity, the assertion then remains true for any pair S_1 and S_2.

There is thus a positive, measurable function $\xi_{S_1}^{S_2} = \xi_{S_1}^{S_2}(C)$ on distributions integrable with respect to $\mu_C^{S_1}$ such that

$$\mu_C^{S_2} = \xi_{S_1}^{S_2}\mu_C^{S_1}.$$

As a consequence, there is a canonical isomorphism $f \to g = \sqrt{\xi_{S_2}^{S_1}}f$ from $L^2(\mu_C^{S_1})$ to $L^2(\mu_C^{S_2})$ that allows us to identify the two spaces and to define $\mathcal{H}_\mathcal{C}$.

Taking, in particular, C to be the unit circle imbedded in the Riemann sphere, we obtain a canonical measure and a canonical L^2-space that we denote $\mathcal{H}$. Then, provided we have parametrized each C_i, we can identify $\mathcal{H}_\mathcal{C}$ with $\otimes_{i=1}^n\mathcal{H}$.

Thus if Σ is any oriented Riemann surface with oriented boundary C the union of $C_l = \cup_{i=1}^m C_i$ and $C_r = \cup_{j=1}^n C'_j$, the first with the given orientation of each C_i, the second with the orientation of each C'_j reversed, then we may identify $\mathcal{H}_{C_l}$ with the tensor product of $\otimes_{i=1}^m\mathcal{H}$ and $\mathcal{H}_{C_r}$ with $\otimes_{j=1}^n\mathcal{H}$. We complete Σ to a closed surface S by parametrizing each C_i and each C'_j and then capping C_l with S_l and C_r with S_r. One possibility is to cap each C_i and C'_j separately, but this is by no means necessary, or even desirable. Let S be the resulting surface. The measure μ_C^S is absolutely continuous with respect to $\mu_{C_l}^S \times \mu_{C_r}^S$. Let

$$\mu_C^S = Z(\phi_l, \phi_r)\mu_{C_l}^S \times \mu_{C_r}^S. \tag{11}$$

Observe that Z is a function of a pair of functions ϕ_l and ϕ_r on distributions that is integrable with respect to $\mu_{C_l}^S \times \mu_{C_r}^S$. So I shall be arguing formally with some fairly fancy notions. There is no difficulty in making them precise for the gaussian measures that arise for free bosons.

The operator K_Σ is to be defined as a map from $L^2(\mu_{C_l}^S)$ to $L^2(\mu_{C_r}^S)$ by means of a kernel. This kernel will necessarily depend on the two measures $\mu_{C_l}^S$ and $\mu_{C_r}^S$, but it will have to be shown that the map from $\mathcal{H}_{C_l}$ to $\mathcal{H}_{C_r}$ does not. The function $Z(\phi_l, \phi_r)$ will be a factor of $K_\Sigma(\phi_l, \phi_r)$ but not the only factor. We shall take

$$K_\Sigma(\phi_l, \phi_r) = \eta_{rl}(\phi_l)Z(\phi_l, \phi_r)\eta_{lr}(\phi_r). \tag{12}$$

The two functions η_{rl} and η_{lr} are still to be defined.

The function η_{lr} is defined just as η_{rl} except that the roles of C_l and C_r are interchanged. So it suffices to define η_{rl}. The curve C_l is the boundary of two Riemann surfaces, S_l and $S'_r = \Sigma \cup S_r$. So $\mu_C^{S_l}$ and $\mu_C^{S'_r}$ are both defined, although, as follows for example from the calculation of (3), they will not be equivalent, in the sense of absolute continuity, to μ_C^S. The curve C_l is, of course, oriented, and we have attached S'_r, also oriented, on the right. The surface $\bar{S}'_r$, obtained by reversing the orientation of S'_r is attached on the left.[6] Although C_l lies in the boundary of S_l and $\bar{S}'_r$, we may again define $\mu_{C_l}^{S_l}$ and $\mu_{C_l}^{\bar{S}'_r}$. Because S_l and S'_r are attached to C_l on the same side, the previous arguments can be extended to compare φ_{S_l} and $\varphi_{\bar{S}'_r}$, defined by $\varphi_{\bar{S}'_r}(\bar{s}) = \bar{\varphi}_{S'_r}(s)$. Here $\bar{s}$ is simply $s \in S_r$ regarded as a point in $\bar{S}'_r$. So we can conclude that $\mu_{C_l}^{S_l}$ and $\mu_{C_l}^{S'_r} = \mu_{C_l}^{\bar{S}'_r}$ are mutually absolutely continuous. Thus we can introduce the Radon-Nikodym derivative $\xi_{S_l}^{S'_r}$ of the second with respect to the first. We define the positive function η_{rl} by the relation

$$\eta_{rl} = \sqrt{\xi_{S_l}^{S'_r}}.$$

The kernel K_Σ is not necessarily bounded. So the most that we can assert at first is that the associated operator is densely defined. This is enough for our present purposes. Indeed I want only to explain why it is well defined projectively, thus up to a constant, independently of the choice of S_l and S_r and why the multiplicative relation $K_{\Sigma'} \circ K_\Sigma = \alpha K_{\Sigma''}$ is valid. It is well to be explicit about the function of the parametrizations, for they are somewhat extrinsic to the constructions. First of all, they allow us to identify $\mathcal{H}_{C_l}$ and $\mathcal{H}_{C_r}$ with tensor products of the space $\mathcal{H}$ with itself. Secondly, the gluing of Σ to Σ' requires an identification of C_r and C'_l and this can be effected by the parametrizations.

The operator K_Σ is defined on $L^2(\mu_C^S)$ as

$$\phi_l \to g(\phi_r) = \int f(\phi_l) K_\Sigma(\phi_l, \phi_r) d\mu_{C_l}^S.$$

Suppose that $\tilde{S}$, defined by $\tilde{S}_l$ and $\tilde{S}_r$, is a second choice for S. Then

$$\tilde{f}(\phi_l) = \sqrt{\xi_{\tilde{S}}^S(C_l)} f(\phi_l)$$

and

$$\tilde{g}(\phi_r) = \sqrt{\xi_{\tilde{S}}^S(C_r)} g(\phi_r).$$

Thus, in terms of $\tilde{f}$ and $\tilde{g}$, the operator K_Σ would be given by a different kernel,

$$\begin{aligned} \tilde{g}(\phi_r) &= \int \tilde{f}(\phi_l) \sqrt{\xi_S^{\tilde{S}}(C_l)} K_\Sigma(\phi_l, \phi_r) \sqrt{\xi_{\tilde{S}}^S(C_r)} d\mu_{C_l}^S \\ &= \int \tilde{f}(\phi_l) \sqrt{\xi_{\tilde{S}}^S(C_l)} K_\Sigma(\phi_l, \phi_r) \sqrt{\xi_{\tilde{S}}^S(C_r)} d\mu_{C_l}^{\tilde{S}}. \end{aligned} \tag{13}$$

We need to verify that the kernel appearing here is equal to that given by the definition (12) applied directly to $\tilde{S}$.

[6] It may be better to think concretely of the unit circle and to take $S_r = \{z \mid |z| \ge 1\}$. Then $\bar{S}_r$ may be identified with $\{z \mid |z| \le 1\}$, the point z on S_r becoming in $\bar{S}_r$ the point $\bar{z}^{-1}$.

As was already observed, the previous arguments that showed that $\mu_C^{S_1}$ and $\mu_C^{S_2}$ were mutually absolutely continuous can be extended to the case that S_1 and S_2 have boundaries and some of the simple closed curves forming C lie in the boundary of both S_1 and S_2. Thus the four functions $\xi_{S_l}^{\tilde{S}_l}(C_l)$, $\xi_{S_r'}^{\tilde{S}_r'}(C_l)$, $\xi_{S_r}^{\tilde{S}_r}(C_r)$, and $\xi_{S_l'}^{\tilde{S}_l'}(C_r)$ are all defined and

$$\xi_S^{\tilde{S}}(C_l) \doteq \xi_{S_l}^{\tilde{S}_l}(C_l)\xi_{S_r'}^{\tilde{S}_r'}(C_l), \quad \xi_S^{\tilde{S}}(C_r) \doteq \xi_{S_l'}^{\tilde{S}_l'}(C_r)\xi_{S_r}^{\tilde{S}_r}(C_r) \tag{14}$$

because, for example,

$$D(\varphi_S) = D(\varphi_{S_l}) + D(\varphi_{S_r'}). \tag{15}$$

The dot above the equalities in (14) indicate that they are only valid projectively. From (15) we can only deduce a relation between gaussian measures up to a constant that will be determined as the quotient of two determinants.

Let $\tilde{Z}(\phi_l,\phi_r)$ be the analogue of $Z(\phi_l,\phi_r)$. Then, as a consequence of (14) and the definition (11),

$$\frac{\tilde{Z}(\phi_l,\phi_r)}{Z(\phi_l,\phi_r)} \doteq \xi_S^{\tilde{S}}(C)\xi_{\tilde{S}_l}^{S_l}(C_l)\xi_{\tilde{S}_r'}^{S_r'}(C_l)\xi_{\tilde{S}_l'}^{S_l'}(C_r)\xi_{\tilde{S}_r}^{S_r}(C_r). \tag{16}$$

The arguments ϕ_l and ϕ_r have been omitted on the right. >From the analogue of (15) for the decompositions $S = S_l \cup \Sigma \cup S_r$ and $\tilde{S} = \tilde{S}_l \cup \Sigma \cup \tilde{S}_r$, we conclude that

$$\xi_S^{\tilde{S}}(C) \doteq \xi_{S_l}^{\tilde{S}_l}(C_l)\xi_{S_r}^{\tilde{S}_r}(C_r). \tag{17}$$

Thus

$$\tilde{Z}(\phi_l,\phi_r) \doteq Z(\phi_l,\phi_r)\xi_{\tilde{S}_r'}^{S_r'}(C_l)\xi_{\tilde{S}_l'}^{S_l'}(C_r) \tag{18}$$

The functions η_{rl} is the square root of $\xi_{S_l}^{S_r'}(C_l)$ and η_{lr} is the square root of $\xi_{S_r}^{S_l'}(C_r)$. Let $\tilde{\eta}_{rl}$ and $\tilde{\eta}_{lr}$ be the analogues of η_{rl} and η_{lr}. They are the square roots of $\xi_{\tilde{S}_l}^{\tilde{S}_r'}(C_l)$ and $\xi_{\tilde{S}_r}^{\tilde{S}_l'}(C_r)$.

For C_l there are four measures in play, those defined by S_l, $\tilde{S}_l$, S_r' and $\tilde{S}_r'$. All are absolutely continuous with respect to each other and there will be obvious relations of transitivity. Making use of (18), we write the kernel of (13) up to a constant factor as

$$\sqrt{\xi_{\tilde{S}_l}^{S_l}(C_l)\xi_{\tilde{S}_r'}^{S_r'}(C_l)}\sqrt{\xi_{S_l}^{S_r'}(C_l)}\frac{1}{\xi_{\tilde{S}_r'}^{S_r'}(C_l)}\tilde{Z}(\phi_l,\phi_r)\sqrt{\xi_{\tilde{S}_l'}^{S_l'}(C_r)\xi_{\tilde{S}_r}^{S_r}(C_r)}\sqrt{\xi_{S_r}^{S_l'}(C_r)}\frac{1}{\xi_{\tilde{S}_l'}^{S_l'}(C_r)}$$

We have to verify that the expression to the left of $\tilde{Z}(\phi_l, \phi_r)$ is equal to $\tilde{\eta}_{rl}$ and that the expression to the right is $\tilde{\eta}_{lr}$. Consider the first and apply the relations of transitivity. The desired relation is immediate.

The most important lesson to be learned from this formal argument is that the relations (14) and the analogous relations for the decompositions $S_l \cup \Sigma \cup S_r$ and $\tilde{S}_l \cup \Sigma \cup \tilde{S}_r$ are critical to the verification that the operator K_Σ is well defined. So we shall need something similar for the Ising model. We first verify the multiplicative property for the free boson. With the freedom we now have in the choice of cappings, this is easy.

After pasting Σ and Σ' together, we cap Σ on the left with S_l and Σ' on the right with S_r to obtain S, a surface that contains both Σ and Σ' and that can be used to define all three operators K_Σ, $K_{\Sigma'}$, and $K_{\Sigma''}$, $\Sigma'' = \Sigma \cup \Sigma'$.

The kernels defining the three operators are given by (12) and its variants for Σ' and Σ'', which we express by adding the appropriate number of primes. When S is chosen as above to be the same for all three of Σ, Σ' and Σ'', then $\eta_{lr}\eta'_{rl} = 1$, $\eta_{rl} = \eta''_{rl}$ and $\eta'_{lr} = \eta''_{lr}$. As a consequence, the multiplicativity becomes

$$\int Z(\phi_l, \phi_m) Z'(\phi_m, \phi_r) d\mu^S_{C_r}(\phi_m) = \alpha Z''(\phi_l, \phi_r) \tag{19}$$

where $C_r = C'_l$ is the curve along which Σ and Σ' are glued.

Consider (19) as a function of ϕ_l for a fixed ϕ_r and multiply this function against the measure $\mu^S_{C_l}$. The result on the right side is a conditional probability of ϕ_l with respect to the measure $\mu^S_{C_l \cup C'_r}$ and the given ϕ_r. The result on the left is the integral over ϕ_m of the conditional probability of ϕ_l with respect to $\mu^S_{C_l \cup C_r}$ and ϕ_m of the conditional probability of ϕ_m with respect to $\mu^S_{C'_l \cup C'_r}$. The Markov property of the gaussian measures, in essence a result of the relations (14) and (15), then yields (19).

4. Possible construction of a conformal field theory. After this cavalier discussion of the free boson, we return to the construction of [I] to see whether it is reasonable to hope that it offers some analogue of (15). It is best to work with the Ising model on a triangular lattice or, if we want to consider models other than translation-invariant planar models, on triangulated surfaces S. The advantage, ultimately of no importance, is that there is no ambiguity about the level curves attached to a given configuration σ. We choose barycenters for each triangle and each edge of the triangulation and join the barycenter of an edge to the barycenters of the two triangles in which it lies. This yields for each edge a broken segment crossing it. The level curve or total, unoriented contour curve attached to a given configuration is a possibly disconnected curve formed as the union of some of these broken segments, those crossing an edge whose two vertices are assigned opposite spins in σ. The elements λ of Λ_σ are obtained by assigning an orientation to each connected component of the level curve. For the triangular lattice or for triangulated surfaces the probability of $\lambda \in \Lambda_\sigma$ is more easily defined than for other models. It is the probability of the configuration σ divided by 2^{l_σ} if l_σ is the number of connected components in the level curve of σ. Thus 2^{l_σ} is also the number of elements in Λ_σ.

I only want to consider the analogue of (14) or (15) at the level of a finite triangulation. There is no question in this paper of proving anything beyond this level or even of pursuing the experiments of [I] further. So let S be closed and compact and let C be a smooth curve, thus the union of a finite number of simple closed parametrized curves, that divides S into two disjoint pieces, S_l and S_r. It is best to suppose – this is clearly a technical issue to be resolved when the time comes to pass to the limit in the mesh – that the level curves cut C transversely. Then each element λ in Λ_σ defines two sets of points on C, the set X of points at which C crosses it with positive orientation and the set Y of points at which C crosses it with negative orientation. Varying σ and choosing for each σ all possible λ, we obtain a collection of pairs (X, Y), possibly repeated and each with a probability, that of λ. The sum of the probabilities is 1. We denote by $\mu_C^\Pi(X, Y)$ the sum over all occurrences of (X, Y) of the probability of the individual occurrence. Then

$$\sum_{(X,Y)} \mu_C^\Pi(X, Y) = 1.$$

As in [I], μ_C^Π may also be considered a measure on distributions. A function f on C is sent by the distribution associated to (X, Y) to the number $\sum f(X) - \sum f(Y)$.

Each vertex σ of the graph Γ is surrounded by a star St_σ that is spanned by the barycenters of the simplices (points, edges and triangles) containing the vertex. Consider the set Π_C of vertices such that St_σ meets C. The set of vertices not in Π_C is divided into two parts, those lying in S_l and those lying in S_r. We denote the two parts by Π_l and Π_r.

The meaning of (14), which we now want to interpret at the level of the finite triangulation, is that the measure μ_C^Π is of the form

$$\xi^{S_l}(C)\xi^{S_r}(C)\nu_C, \tag{20}$$

where ν_C is a measure that is independent of the choice of S_l and S_r and where, for example, $\xi^{S_l}(C)$ is a function of (X, Y) that depends only on S_l but not on S_r. In fact, it will depend on the collection

Θ_l of edges connecting points of Π_l to other points of Π_l or to points in Π_C and the graph they define. Given (20), we could introduce the Radon-Nikodym derivatives

$$\xi^{S_l}_{S'_l}(C) = \frac{\xi^{S_l}(C)}{\xi^{S'_l}(C)}$$

and the essential factors η_{rl} and η_{lr}. There would be, by the way, nothing canonical about such a factorization since ν_C is clearly not uniquely determined. Different ν lead, however, to the same Radon-Nikodym derivatives. The measure ν may have no limit as the mesh decreases to 0, so that in the limit, when the mesh becomes zero, it is purely fictitious and, as for the free boson, only the Radon-Nikodym derivatives survive. For the Ising model there appears to be an additional complication, so that we cannot simply take the definition (12) at the finite level and then pass to the limit. Before explaining the difficulty, it is useful, as a supplement to the discussion of the free boson, to take a few lines to explain to what the definition (12) reduces at the finite level when (20) is available, not only for a curve C that divides S into a left part S_l and a right part S_r but also for a curve $C = C_l \cup C_r$ that separates S into three parts S_l, Σ and S_r as in the definition of the operator K_Σ.

For such a curve, the analogue of (20) takes the form

$$\mu^\Pi_C = \xi^{S_l}(C_l)\xi^\Sigma \xi^{S_r}(C_r)\nu_{C_l} \times \nu_{C_r}.$$

The formula (20) applied to C_l and C_r yields

$$\mu^\Pi_{C_l} = \xi^{S_l}(C_l)\xi^{S'_r}(C_l)\nu_{C_l}, \qquad \mu^\Pi_{C_r} = \xi^{S'_l}(C_r)\xi^{S_r}(C_r)\nu_{C_r}.$$

The map

$$f \to F = f\sqrt{\xi^{S_l}(C_l)}\sqrt{\xi^{S'_r}(C_l)}$$

identifies $\mathcal{H}_{C_l} = L^2(\mu^\Pi_{C_l})$ with $L^2(\nu_{C_l})$. The map

$$g \to G = g\sqrt{\xi^{S'_l}(C_r)}\sqrt{\xi^{S_r}(C_r)}$$

identifies $\mathcal{H}_{C_r} = L^2(\mu^\Pi_{C_r})$ with $L^2(\nu_{C_r})$.

The kernel K_Σ defined by (12) is

$$\text{(21)} \qquad \sqrt{\frac{\xi^{S'_r}(C_l)}{\xi^{S_l}(C_l)}}\frac{\xi^{S_l}(C_l)\xi^\Sigma\xi^{S_r}(C_r)}{\xi^{S_l}(C_l)\xi^{S'_r}(C_l)\xi^{S'_l}(C_r)\xi^{S_r}(C_r)}\sqrt{\frac{\xi^{S'_l}(C_r)}{\xi^{S_r}(C_r)}}.$$

We have suppressed the two variables ϕ_l and ϕ_r. Integrated against f with respect to the measure $\mu^\Pi_{C_l}$, this kernel yields the image g of f under the operator K_Σ. The argument of f is ϕ_l, that of g is ϕ_r.

Expressed in terms of F and G and an integration with respect to ν_{C_l}, the kernel of K_Σ is the product of (21) with

$$\frac{1}{\sqrt{\xi^{S_l}(C_l)\xi^{S_r'}(C_l)}}\xi^{S_l}(C_l)\xi^{S_r'}(C_l)\sqrt{\xi^{S_l'}(C_r)\xi^{S_r}(C_r)}$$

When all possible cancellations in the product are carried out, nothing is left but ξ^Σ. The multiplicative property then reduces to a relation similar to (19),

$$\int \xi^\Sigma(\phi_l,\phi)\xi^{\Sigma'}(\phi,\phi_r)d\nu_{C_r}(\phi) = \alpha\xi^{\Sigma''}.$$

We return to the Ising model and the apparent failure of (20). Given (X, Y), the σ that generate it are defined by a triple: σ_l on the vertices of Π_l, σ_r on the vertices of Π_r, and σ_C on the vertices of Π_C. The configuration σ is determined up to a global sign on each component of S by the level curve λ. The Boltzmann weight is not affected by these global signs. That factor β_C of the Boltzmann weight for σ contributed by edges joining vertices in Π_C is determined by (X, Y) alone. The Boltzmann weight is a product of this factor and two other factors, β_l and β_r, the first the product of the contributions from the edges in Θ_l crossing that part λ_l of the level curve that lies in S_l, the second the product of the contributions from the edges in Θ_r that cross λ_r. The number l_σ of connected closed curves in λ is given by $l_\sigma = l_l + l_r + l_C$, where l_l is the number of closed curves lying entirely in S_l, l_r the number lying entirely in S_r, and l_C the number that meet C. So, apart from the normalizing factor given by the partition function,

$$\mu_C^\Pi(X,Y) = 2^{|S|}\beta_C\sum_{\lambda_l}\sum_{\lambda_r}\frac{\beta_l\beta_r}{2^{l_l+l_r+l_C}}, \tag{22}$$

the sum being over all possible λ_l and λ_r compatible with the given collection (X, Y) of positively and negatively oriented crossings of C. The exponent $|S|$ is the number of connected components of S.

It is the term 2^{l_C} in the denominator of (22) that prevents the factorization of (20). So we might attempt to modify the construction. There are two pairings of X with Y associated to compatible λ and (X, Y),. If, as is implicit in the notation, we have been careful with our orientations on C, then at each $x_i \in X$, $i = 1,\ldots,N$ one component of λ crosses from S_r into S_l. Let y_i be the first point at which it crosses back into S_r. The first pairing is $W_l = \{(x_1, y_1),\ldots,(x_N, y_N)\}$. The second, $W_r = \{(y_1', x_1'),\ldots,(y_N', x_N')\}$, is defined in the same way, except that the roles of S_l and S_r are reversed. These two collections together define a distribution on $C \times C$,

$$f \to \sum_i f(x_i, y_i) - \sum_i f(y_i', x_i'),$$

f being a function on $C \times C$. Taking $f(x, y) = g(x) - g(y)$, we recover twice the original distribution,

$$f \to \sum \{g(x_i) - g(y_i) - g(y'_i) + g(x'_i)\} = 2 \sum_i \{g(x_i) - g(y_i)\}$$

because $\{x_i\} = \{x'_i\}$ and $\{y_i\} = \{y'_i\}$. This construction does not demand the global existence of S_l and S_r; it simply requires that C be oriented.

The measure on the collection Λ of all possible level curves defines one, $\mu^{\Pi}_{C\times C}$, on the family of collections

$$W = (W_l, W_r) = (\{(x_1, y_1), \dots, (x_N, y_N)\}, \{(y'_1, x'_1), \dots, (y'_N, x'_N)\}) \tag{23}$$

associated to the λ in it. To calculate $\mu^{\Pi}_{C\times C}(W)$, we first construct λ_l in S_l compatible with the first component W_l of W, and λ_r in S_r compatible with the second, W_r. If we join them at the points where they meet on C, they form together a λ compatible with W and all such λ are so obtained. Once the collection λ of oriented level curves is fixed, the configuration σ is determined up to the choices of global sign. Since l_C is determined by W alone, we write $l_C = l_W$. The relation (22) now becomes

$$\mu^{\Pi}_{C\times C}(W) = 2^{|S|-l_W} \beta_C \{\sum_{\lambda_l} \frac{\beta_l}{2^{l_l}}\}\{\sum_{\lambda_r} \frac{\beta_r}{2^{l_r}}\}, \tag{24}$$

$|\lambda_l|$ and $|\lambda_r|$ being the number of closed curves in λ_l and λ_r respectively, and l_W being the number of closed curves in λ that meet C. It is a number that is determined by W alone. The formula (20) for the measure on the collection of W follows from (24).

Unfortunately measures on distributions on $C \times C$, and this is what might result from (24) on passage to the limit over decreasing mesh, have a number of disadvantages that make them unsuitable for the construction of a conformally invariant theory. For example, there is no possibility that when C has more than one connected component the absolute continuity class of $\mu^{S}_{C\times C}$ is independent of S either at the finite level or in the limit. So we have to find our way back to pairs (X, Y).

The object W is a pair (W_l, W_r) and there are maps $W_l \to (X, Y)$ and $W_r \to (X, Y)$. Consider the matrix with entries $a(W_l, W_r) = 2^{-l_W}$. If the entries could be written as

$$a(W_l, W_r) = \alpha b(W_l) b(W_r), \tag{25}$$

α a constant, then the right side of (24) would become

$$2^{|S|} \alpha \beta_C \{\sum_{\lambda_l} \frac{b(W_l)\beta_l}{2^{l_l}}\}\{\sum_{\lambda_r} \frac{b(W_r)\beta_r}{2^{l_r}}\}.$$

Passage to the level of (X, Y) requires summing over all pairs (W_l, W_r) for which the image of both W_l and W_r is (X, Y). The mass of (X, Y), again apart from the normalization given by the partition function, would be

$$\mu_C^{\Pi}(X,Y) = 2^{|S|}\alpha\beta_C\{\sum_{\lambda_l, W_l \to (X,Y)} \frac{b(W_l)\beta_l}{2^{l_l}}\}\{\sum_{\lambda_r, W_r \to (X,Y)} \frac{b(W_r)\beta_r}{2^{l_r}}\}. \tag{26}$$

This is the factorization required.

There is no possibility that the simple representation (25) of $a(W_l, W_r)$ exists, but as we are passing to a limit so much is not required. The relation (25) has presumably only to hold approximately except for a collection of (X, Y) whose measure tends to zero as the mesh does. This statement will be easier to understand once we examine the matrix $A(X, Y) = (a(W_l, W_r))$ more carefully. It depends on (X, Y), X and Y being two sets of points with the same number n of elements. We label them both as $\{1, \dots, n\}$. It is likely that the probability that n is less than any given bound is going to zero with the mesh. The family W_l is nothing but a permutation r of $\{1, \dots, n\}$ and W_r a permutation s^{-1}. The number l_W is the number $\gamma(rs^{-1})$ of cycles in rs^{-1}. Thus $A(X, Y)$ is the matrix of

$$R = R_n = \sum_r 2^{-\gamma(r)} r \tag{27}$$

in the regular representation of the symmetric group on n symbols with its standard basis.

We can decompose the regular representation into irreducible constituents τ. Since (27) is clearly a central element in the group algebra, it is represented as a scalar matrix in each of these constituents. If absolute precision is not demanded, then the approximate form of (25) is that, as n increases, the eigenvalues of $\tau(R)$, all of which are equal, divided by the eigenvalue for the trivial representation approach zero, provided that τ is not trivial. The vector $(b(W))$ would then have all its components equal. As a somewhat unexpected conclusion to a lecture on percolation and the Ising model, I therefore briefly describe the necessary representation theory of the symmetric group, which I take from [J].

5. Calculations for the symmetric group. We begin with the calculation of $\tau(R)$ for the trivial representation. It is given by a generating function, the coefficient of x^n being the eigenvalue for the symmetric group on n symbols divided by $n!$. For a given n we give the cycle lengths as i_j cycles of length j, for $j = 1, 2, \dots$, with $\sum_j i_j j = n$. The number of cycles corresponding to these lengths is

$$n! \prod_{j=1}^{\infty} (\frac{(j-1)!}{j!})^{i_j} \frac{1}{i_j!} = n! \prod_{j=1}^{\infty} (\frac{1}{j})^{i_j} \frac{1}{i_j!}.$$

Observe that when $i_j = 0$ the corresponding factor is 1, so that it is permissible to write an infinite product. It is to be multiplied by

$$\prod_j \frac{1}{2^{i_j}}.$$

Multiplying by x^n, dividing by $n!$, and summing over all possible families of nonnegative integers i_j, we obtain for the generating function the result

$$\prod_{j=0}^{\infty} \exp(x^j/2j) = \exp(-\ln(1-x)/2) = \frac{1}{(1-x)^{1/2}}.$$

Thus the eigenvalue of R_n for the trivial representation is

(28).
$$\frac{1}{2}\frac{3}{2}\dots(n-\frac{1}{2}).$$

We can expect that this is the largest eigenvalue of R_n.

A similar calculation for the one-dimensional representation $\tau(r) = \mathrm{sgn}(r)$, but with division by $(-1)^n n!$, yields the series of $(1-x)^{1/2}$. The quotient of the eigenvalues of R_n in the two representations is thus

$$(-1)^{n-1}\frac{\frac{1}{2}\frac{1}{2}\frac{3}{2}\dots\frac{2n-3}{2}}{\frac{1}{2}\frac{3}{2}\dots\frac{2n-1}{2}} = \frac{(-1)^{n-1}}{2n-1},$$

confirming our hopes.

The general irreducible representation τ is given as τ^μ, where μ is a partition of the set $\{1,\dots,n\}$. The properties of τ^μ are described in [J]. First of all, there is an ordering on partitions, the partition $\lambda = \{l_1 \geq l_2 \dots\}$ dominating the partition $\mu = \{k_1 \geq k_2 \geq \dots\}$ if and only if $\sum_{i=1}^j l_i \geq \sum_{i=1}^j k_i$ for all j. The representations τ^μ have the property that τ^μ is contained exactly once in the representation ι^μ induced from the trivial representation of the subgroup fixing the partition μ and every other representation contained in this induced representation is equivalent to a τ^λ, $\lambda \geq \mu$.

The trace of $\iota^\mu(R_n)$ is readily calculated. Let $\mu = \{k_1,\dots,k_s\}$. Then the trace of $\iota^\mu(R)$ is obtained by taking the sum over all decompositions of $\{1,\dots,n\}$ into s subsets of respectively $k_1,\dots,k_s$ elements of the product of the traces of R_{k_l} with respect to the trivial representation of the symmetric group on k_l elements, thus

$$\mathrm{tr}(\iota^\mu(R_n)) = \frac{n!}{k_1!\dots k_s!}\prod_{j=1}^{s}\frac{1}{2}\frac{3}{2}\dots(k_j-\frac{1}{2}).$$

To compute the trace of τ^μ, we use the *determinantal form* appearing on p. 74 of [J]. Writing formally $\iota^\mu = [k_1]\ldots[k_s]$ when $\mu = \{k_1, \ldots, k_s\}$, we can treat any formal determinant

$$\big|\,[m_{i,j}]\,\big| \tag{29}$$

that is of size $s \times s$ and in which $\sum_i m_{i,r(i)} = n$ is independent of the permutation r as a linear combination of induced representations,

$$\sum_r \mathrm{sgn}(r)[m_{1,r(1)}][m_{2,r(2)}]\ldots[m_{s,r(s)}].$$

If in (29) $m_{i,j} = 0$, then $[m_{i,j}]$ is a multiplicative identity, whereas if $m_{i,j} < 0$ then $[m_{i,j}]$ is the multiplicative zero.

The determinantal form described and proved in [J] expresses τ^μ in this way. If μ is given by $k_1 \geq k_2 \geq \ldots \geq k_s$, then

$$\tau^\mu = \big|\,[k_i - i + j]\,\big|. \tag{30}$$

Since the dimension of ι^μ, $\mu = \{k_1, \ldots, k_s\}$, $n = \sum k_i$, is $n!/k_1!\ldots k_s!$, this yields a simple formula for the dimension of τ^μ. It is

$$n!\left|\frac{1}{(k_i - i + j)}\right|. \tag{31}$$

It also yields a simple formula for the trace of $\tau^\mu(R_n)$ as

$$n!\left|\frac{\frac{1}{2}\frac{3}{2}\ldots(k_i - i + j - \frac{1}{2})}{(k_i - i + j)!}\right|.$$

The eigenvalues of $\tau^\mu(R_n)$ are calculated as the quotient

$$\frac{\left|\frac{\frac{1}{2}\frac{3}{2}\ldots(k_i-i+j-\frac{1}{2})}{(k_i-i+j)!}\right|}{\left|\frac{1}{(k_i-i+j)}\right|}. \tag{32}$$

Factorials of negative numbers are of course infinite. Moreover, the entry in the determinant of the numerator is to be taken to be 1 when $k - i + j = 0$.

For a partition consisting of a single term, this clearly agrees with our previous formula. As an additional confirmation, take the simple partition of n given by $\mu = \{1, 1, \ldots, 1\}$. Then the determinant of (31) is

$$\begin{vmatrix} 1 & \frac{1}{2!} & \frac{1}{3!} & \frac{1}{4!} & . & . & . \\ 1 & 1 & \frac{1}{2!} & \frac{1}{3!} & . & . & . \\ 0 & 1 & 1 & \frac{1}{2!} & . & . & . \\ 0 & 0 & 1 & 1 & . & . & . \\ . & . & . & . & . & . & . \end{vmatrix}$$

Multiplying this determinant by

$$1 = \begin{vmatrix} 1 & 0 & 0 & 0 & 0 & . & . & . \\ -1 & 1 & 0 & 0 & 0 & . & . & . \\ 2! & -2! & 1 & 0 & 0 & . & . & . \\ -3! & 3! & -\frac{3!}{2} & 1 & 0 & . & . & . \\ 4! & -4! & \frac{4!}{2!} & -\frac{4!}{3!} & 1 & . & . & . \\ . & . & . & . & . & . & . & . \end{vmatrix},$$

we obtain the determinant of an upper-diagonal matrix with entries $1, 1/2, 1/3, \ldots$ along the diagonal. So the formula (31) yields 1. The determinant in the numerator of (32) is

$$\begin{vmatrix} \frac{1}{2} & \frac{\frac{1}{2}\frac{3}{2}}{2!} & \frac{\frac{1}{2}\frac{3}{2}\frac{5}{2}}{3!} & \frac{\frac{1}{2}\frac{3}{2}\frac{5}{2}\frac{7}{2}}{4!} & . & . & . \\ 1 & \frac{1}{2} & \frac{\frac{1}{2}\frac{3}{2}}{2!} & \frac{\frac{1}{2}\frac{3}{2}\frac{5}{2}}{3!} & . & . & . \\ 0 & 1 & \frac{1}{2} & \frac{\frac{1}{2}\frac{3}{2}}{2!} & . & . & . \\ 0 & 0 & 1 & \frac{1}{2} & . & . & . \\ . & . & . & . & . & . & . \end{vmatrix}$$

If it is multiplied by

$$1 = \begin{vmatrix} 1 & 0 & 0 & 0 & 0 & 0 & . & . & . \\ -2 & 1 & 0 & 0 & 0 & 0 & . & . & . \\ -8 & 4 & 1 & 0 & 0 & 0 & . & . & . \\ -16 & 8 & 2 & 1 & 0 & 0 & . & . & . \\ -\frac{128}{5} & \frac{64}{5} & \frac{16}{5} & \frac{8}{5} & 1 & 0 & . & . & . \\ . & . & . & . & . & . & . & . & . \end{vmatrix}$$

the result is again an upper-diagonal matrix but now with entries $\frac{1}{2}, -\frac{1}{2}\frac{1}{2}, -\frac{3}{2}\frac{1}{3}, -\frac{5}{2}\frac{1}{4}, \ldots$ along the diagonal. To pass from one row of the matrix to the next we multiply the nonzero entries by $2k/(2k-3)$, add one entry equal to 1, and the necessary zeros.

Suppose $\mu = \{k_1, k_2\}$, $k_1 \geq k_2$, $k_1 + k_2 = n$. Then the denominator of (32) is equal to

$$\frac{1}{k_1!k_2!}(1 - \frac{k_2}{k_1 + 1}).$$

The numerator is equal to

$$\frac{\frac{1}{2}\ldots(k_1 - \frac{1}{2})}{k_1!}\frac{\frac{1}{2}\ldots(k_2 - \frac{1}{2})}{k_2!}(1 - \frac{k_2(k_1 + \frac{1}{2})}{(k_2 - \frac{1}{2})(k_1 + 1)}).$$

The quotient is to be divided by (28). This yields

$$\frac{\frac{1}{2}\dots(k_1-\frac{1}{2})\frac{1}{2}\dots(k_2-\frac{1}{2})}{\frac{1}{2}\dots(n-\frac{1}{2})}(1-\frac{k_2(k_1+\frac{1}{2})}{(k_2-\frac{1}{2})(k_1+1)})\Big/(1-\frac{k_2}{k_1+1})$$

The second term of the numerator is universally bounded and the denominator is at least $(k_1-k_2+1)/k_1$. The first term of the terminator is bounded by a universal constant times

$$\frac{\Gamma(k_1+\frac{1}{2})\Gamma(k_1+\frac{1}{2})}{\Gamma(N+\frac{1}{2})}=O((\frac{k_1}{n})^{k_1}(\frac{k_2}{n})^{k_2}),$$

the constant implicit in this relation being universal. The quotient clearly goes to 0 as n approaches infinity, independently of the ratio of $k_1/k_2>1$. I have not yet tried to establish something similar for arbitrary $k_1\geq\ldots\geq k_s$. Experiments can, however, begin without a full understanding of the eigenvalues of $\tau(R_n)$.

6. Final remarks. Even if the constructions of §4 have something to offer, it is not clear how to set about convincing oneself that they can indeed be made, thus that the pertinent scaling limits exist, or that the intuitive arguments that we sketched can be rendered effective. For experiments, one might begin with models on the Riemann sphere or, by conformal invariance, with translation-invariant models in the plane or on a cylinder. The construction is also possible for percolation, and for percolation experiments will be easier to perform.

Neither for percolation nor for the Ising model do I see at all clearly what information might be contained in the operators K_Σ. If they can be defined, it is more than likely, indeed almost certain, that they contain not only the expected unitary conformal field theories but also nonunitary ones.

Even if experiments establish that the reflections of this paper are well founded, the problem of proving that the scaling limits exist will remain. There may be some analogue of the finite-model for percolation described in [P3], although its definition will have to be more elaborate, not alone but in part because there is a dependence in the Ising model not present in percolation. Crossing probabilities for regions that do not overlap are independent in percolation. The measures μ_C will, on the contrary, change even when the conditioning data are taken from outside C, although the influence diminishes with increasing distance. It can be taken into account, but whether it can be taken into account effectively so that dynamical systems of manageable size result is another matter. For the construction of a conformal field theory, it may very well be better to work directly with definitions based on (26), and for finite models it may be best to begin with the factorization given by that formula.

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