Review of

$SL_2(\mathbf{R})$ by Serge Lang

Given the formalism of quantum mechanics, the study of those of its laws which are invariant under the Lorentz group inevitably leads to infinite-dimensional representations of both the homogeneous and inhomogeneous forms of the group. Responding to earlier work of Dirac and Wigner, Bargmann, Gelfand-Naĭmark, and Harish-Chandra published in 1946 and 1947 classifications of the unitary representations of the homogeneous Lorentz group, or rather of a covering group, $SL(2, \mathbf{C})$. Bargmann, because he was interested in the representations of the inhomogeneous group, was led as well to classify the representations of $SL(2, \mathbf{R})$, a covering group of the Lorentz group in three variables, two in space and one in time.

From these innocent beginnings the mathematical theory of infinite-dimensional representations has expanded relentlessly, forgetting its origins in physics but encroaching on other domains of mathematics, especially number theory, to which its methods, those of functional analysis with a heavy admixture of Lie theory, have been foreign. Since the training of many contemporary number-theorists has been primarily algebraic, even those who view the new methods with favor find them difficult to assimilate. Some simple introductions are needed, not so much to expose the techniques, or even the basic concepts, but just to pierce the tough rind of unfamiliarity. Such is the purpose of $SL_2(\mathbf{R})$. It is a rough-hewn book, leisurely and informal, which in the manner of a good graduate course, conscientiously explains the heterogeneous facts from various domains which could be stumbling blocks for the novice, and may be exactly what is needed.

Bargmann, to whose paper many later students have turned for an introduction to the subject, discovered in particular a discrete series of irreducible representations of $SL(2, \mathbf{R})$ with square-integrable matrix coefficients. It is remarkable that most of the phenomena which are significant for the general theory, not only the discrete series, whose importance cannot be exaggerated, but also other things more easily overlooked, appear already in $SL(2, \mathbf{R})$; those who are led through it by an experienced guide will, if they later penetrate the general theory, meet nothing totally unfamiliar.

But one does not reach new continents by skirting the coasts of home. The physicist is concerned almost exclusively with the internal structure of the irreducible representations, for example with the eigenvalues of the angular momenta and other operators of physical interest, but the mathematician is

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not. It does no harm to know it and it must occasionally be brought into play, but it is the harmonic analysis which matters to him, the characters of irreducible representations and the decomposition of distributions on the group invariant under inner automorphisms into linear combinations of characters. Because the transition to the traditional viewpoint in automorphic forms is through the internal structure and because the still immature representation theory of groups over nonarchimedean fields needs it as a prop, there is danger, particularly severe for the algebraist since it belongs to a category of thought with which he feels at ease, that the internal structure will be emphasized at the expense of the harmonic analysis, and the point be missed. Over nonarchimedean fields the harmonic analysis must of course be supplemented by the arithmetic, but that is some else and not pertinent here.

The internal structure of the irreducible representations of $SL(2, \mathbf{R})$, especially the decomposition with respect to the action of the group of proper rotations, is so simple that it is worth learning, if only to know the thing one's neighbor knows. Lang describes it of course, and treats the representations induced from a parabolic subgroup, a simple but essential construction. Only the spherical functions bi-invariant under the group of proper rotations are introduced, and the role of the differential equations in determining their asymptotic behaviour, central to the later work Harish-Chandra on the harmonic analysis, is not described. The asymptotic behaviour is analyzed using integral formulas, which have not proved very useful in general, rather than Harish-Chandra's techniques, which for $SL(2, \mathbf{R})$ become elementary, reducing to the method of Frobenius. Lang confines his discussion of the harmonic analysis to the very important Plancherel formula, the explicit representation of the distribution $f \to f(1)$ as an integral of characters.

But it could have been treated more fully, in order to bring its outlines into sharper focus. For the analyst, one might introduce the Schwartz space and the notion of a tempered distribution, prove that the characters are functions, fairly easy to do for $SL(2, \mathbf{R})$ and certainly enlightening, and perhaps also show how to prove the existence of the characters of the discrete series without explicitly realizing the representations. There is some misunderstanding even among experts about the form taken by Harish-Chandra's proofs in the simple case of $SL(2, \mathbf{R})$. Since they may soon be superseded by others, it would be useful to examine them carefully before it is too late.

To guide the arithmetician one can stress the integrals over conjugacy classes, characterizing the Harish-Chandra transform of functions in the Schwartz space of smooth, compactly supported functions and, above all, drawing attention to the Selberg principle and the orthogonality relations for the characters of the discrete series. To fix the harmonic analysis in the minds of both, one could discuss the trace formula for discrete subgroups Γ of $SL(2, \mathbf{R})$ with compact quotient. If one does not treat the adelic groups the best arithmetical applications are precluded, but the geometrical applications, especially those which bring into play zeta-functions associated by Selberg to surfaces of constant negative curvature, are available, of wide appeal, and not difficult.

It is hard for anyone who is not a specialist to place the trace formula in perspective. If the quotient is compact, it is just a clever reformulation of facts with which we are all familiar from the study of induced characters for finite groups and is easily verified; if it is not, the cusps, especially in higher dimensions, entail great technical complications. Yet it may be asserted, without gross exaggeration that to a number-theorist groups with compact quotient are just as important as the others. The theorems which can be proved for them with the aid of the trace formula are equally striking. The statements are analogous to those for noncompact quotients and the same methods are used, but the essential ideas, freed of technical encumbrances, become patent. Out of obedience to old habits of thought, and occasionally for arcane motives of convenience, it is usually for noncompact quotients that the theorems are stated and proved. An obscure exception is described by the present reviewer in a lecture, *Shimura varieties and the Selberg trace formula*, which appears in the proceedings of the 1975 U.S.-Japan Seminar on Number Theory, held in Ann Arbor. Since the complications arising from the cusps were even more formidable than usual, and could be overcome only with the introduction of much additional machinery, he was driven by desperation to resist tradition's tyranny.

The difficulties for groups with a quotient which is not compact appear because the representation ρ of $G = SL(2, \mathbf{R})$ on the space $L^2(\Gamma\backslash G)$ of square-integrable functions on $\Gamma\backslash G$ does not decompose into a direct sum of irreducible representations and, even when the function f is smooth and of compact support, the operator $\rho(f) = \int_G f(g)\rho(g)dg$ is not necessarily of trace class. The theory can only begin after one removes from $\rho(f)$ its projection on the continuous spectrum. The continuous spectrum is analyzed by means of the Eisenstein series, so called because they reduce for special values of the parameters to the Eisenstein series appearing in the study of elliptic functions. The series depend on a complex parameter s, as well as another parameter which is less important and may be suppressed here. However the series converges only in a half-plane $\mathrm{Re}\ s > c > 0$, and it is its value on the imaginary axis that is needed; so analytic continuation is called for. Selberg's investigations revealed that the continuation could be effected very deftly – a twist, a flick of the wrist, and the prize is won. All that is needed are elementary facts about selfadjoint operators, not even the spectral theorem, and about the geometry of fundamental domains. Since his method is fitted to the problem, it is easily formulated in representation-theoretic and adelic terms, whereby it becomes even simpler, and yields all that is needed for the trace formula, in the form suitable to arithmetical applications. Among other things it throws the intertwining operators, which play an important role in the local theory as well, into relief. Selberg's proofs were never published or even made widely available. Since they were intrinsic to the problem they could be used to push a good way into the general theory, becoming thereby so entangled with the theory of algebraic groups that their simplicity was hidden, and their nature often misunderstood. It is not completely revealed by Kubota's nonetheless useful account of the original method in his *Elementary theory of Eisenstein series*.

There are two other methods for dealing with Eisenstein series and the continuous spectrum, but they introduce elements foreign to the problem and so far have been of limited applicability. This notwithstanding, they are the methods used in $SL_2(\mathbf{R})$. The simplest starts from the Poisson summation formula and has, in various guises, been with us for a long time. It works well for subgroups of $SL(2, \mathbf{Z})$, and perhaps for subgroups of $SL(n, \mathbf{Z})$ too. Its limitations are recognized, and Lang employs it only for the sake of a quick introduction.

The other method is newer and appeared only after the problem of the analytic continuation of Eisenstein series had been solved for general groups. It has exercised a strange attraction on a number of mathematicians, Lang among them, and acquired somehow a reputation of being more analytic. It is in fact not unrelated to Selberg's method, but this flows easily along a natural course, while that moves through a channel cut by the machinery of perturbation or, more precisely, scattering theory. Scattering theory, to which Faddeev has written an enlightening introduction (translated in J. Mathematical Phys., 1963), is important for its own sake, and may be a useful weapon for the number-theorist, and the rest of us too; if not for use against the Eisenstein series which have, after all, already surrendered, then against stronger, more stubborn foes; so we can be grateful to Lang for pressing it into our hands. Moreover, since it has been easy to forget that, like everything else, Selberg's method had antecedents, it is instructive to place it alongside the methods arising from scattering theory and to note the fraternal likeness. But this is of interest only to the initiated. The beginner should be shown an easy path, free of red herring and leading to some outstanding problems, which for the Eisenstein series are usually in higher dimensions and primarily arithmetic, concerned not only with the analytic continuation which is known but with the location of the poles contributing to the spectrum. Their solution probably demands a better understanding of the Euler products associated to automorphic forms and of the intertwining operators and their normalizations.

But we should not forget the purpose of the book, which was not intended to teach the reader everything about $SL(2, \mathbf{R})$. It is written by an outsider, although not to mathematics or to exposition, for outsiders, and in consulting his own needs he has probably met theirs. $SL_2(\mathbf{R})$, which introduces the harmonic analysis through the Plancherel formula and the analytic theory of automorphic forms through the Eisenstein series, may take its place alongside the author's other books, which for many of us have been the entrance to topics that could otherwise have remained inaccessible.