The Dirac Monopole and Induced Representations *

In this note a mathematically transparent treatment of the Dirac monopole is given from the point of view of induced representations. Among other things the question of bound states for the spinning electron in the field of a magnetic monopole is considered.

In 1948, as he was turning from theoretical physics to mathematics, Harish-Chandra wrote one last paper [**5**] on a physical topic, investigating a question inspired by Dirac, and perhaps even proposed by him. Does an electron moving according to Dirac's equation in the field of a magnetic monopole have a bound state? The technique involved, namely, separation of variables, is of course elementary, and would reappear but at a much deeper level repeatedly in Harish-Chandra's later work on harmonic analysis on semisimple groups. However, this time, his mind on other matters, he handled it perfunctorily, and went astray with the calculations, concluding – incorrectly – that there could be no bound states. The matter has since been dealt with correctly ([**4**], [**6**], [**8**]). However, the derivation of the radial equations in [**6**] and [**8**] is not so efficient as it might be. In view of the possible historical interest to students of Harish-Chandra's later work, a brief, mathematically transparent treatment does not seem out of place. It is the purpose of this note, which does not touch on any questions of serious current interest, to provide it.

In [**2**] Dirac considers the wave function ψ of a charged particle without spin, observing that one can replace ψ by $e^{i\beta}\psi$ without changing the distribution $|\psi|^2$, which is what counts. The function β depends on the coordinates g, x^1, x^2, x^3. The substitution replaces

$$\frac{1}{i}\frac{\partial}{\partial t} \text{ by } \frac{1}{i}\frac{\partial}{\partial t} + \frac{\partial \beta}{\partial t}$$

and

$$\frac{1}{i}\frac{\partial}{\partial x^j} \text{ by } \frac{1}{i}\frac{\partial}{\partial x^j} + \frac{\partial \beta}{\partial x^j},$$

but, since

$$\frac{1}{i}\frac{\partial}{\partial t} \text{ and } \frac{1}{i}\frac{\partial}{\partial x^j}$$

do not appear alone but in the combinations

$$\frac{1}{i}\frac{\partial}{\partial t} + A_0, \frac{1}{i}\frac{\partial}{\partial x^j} + A_j,$$

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this effect can be countered by a change in the electromagnetic potentials A_0, A_1, A_2, A_3. The upshot is that ψ can be regarded as a section of a line bundle with metric and with a connection defined by the electromagnetic potentials.

The pertinent line bundle is on M, four-dimensional space with the line $x^1 = x^2 = x^3 = 0$ removed. This space is to be identified with $\mathbf{R} \times \mathbf{R}^+ \times H\backslash G$, where G is $SU(2)$ and H is the group of diagonal matrices. As usual G is mapped to $SO(3)$ and thus acts on 3-space $((x^1, x^2, x^3) \to (y^1, y^2, y^3) = (x^1, x^2, x^3)A(g))$ in such a way that the Pauli matrices

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

multiplied by $i/2$ correspond to infinitesimal rotations through the x^1, x^2, and x^3-axes. Thus

$$\begin{pmatrix} y^3 & y^1 - iy^2 \\ y^1 + iy^2 & -y^3 \end{pmatrix} = f^{-1} \begin{pmatrix} x^3 & x^1 - ix^2 \\ x^1 + ix^2 & -x^3 \end{pmatrix} g.$$

Then $(r, g) \to r(001)A(g)$ identifies $\mathbf{R}^+ \times H\backslash G$ with 3-space minus the origin. The line bundle is defined by a one-dimensional representation

$$\rho : \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{i\theta} \end{pmatrix} \to e^{in\theta}$$

of H, sections being complex functions f on $\mathbf{R} \times \mathbf{R}^+ \times G$ satisfying

$$f(t, r, hg) = \rho(h) f(t, r, g).$$

The connection, like the bundle, is a product trivial on the first two factors. Let $\mathcal{H}^\perp$ be the orthogonal complement of $\mathcal{H}$ in G and regard functions on $H\backslash G$ as H-invariant functions on G. We prescribe that for $X \in \mathcal{H}^\perp$ the tangent vector X_g at Hg defined by

$$X_g f = \frac{d}{dt} f(\exp tX \cdot g)|_{t=0}$$

acts on sections by the same formula, and verify that this yields a well-defined connection.

The curvature of this connection can be computed on $H\backslash G$ and, since $[\mathcal{H}^\perp, \mathcal{H}^\perp] \subseteq \mathcal{H}$, is a two-form that takes the value $\frac{1}{2}\rho([X^1, X^2])$ at $X_g^1 \wedge X_g^2$. Taking $X^1 = i\sigma^1/2, X^2 = i\sigma^2/2$ and recalling that when divided by $-i/2$ the curvature on M yields a two-form that gives the electric and magnetic fields, we see that the associated electric field is 0 and the magnetic field purely radial and equal to

$$-\frac{n}{2r^3}(x^1, x^2, x^3).$$

Observe that all constructions are invariant under the action of G on $H\backslash G$, which of course yields the usual action of G on M. Consequently the magnetic field is spherically symmetric and need only be evaluated at $(0,0,r)$. Notice also that at a point (r,g) where $g=1$,

$$X_1^1 = -r\frac{\partial}{\partial x^2},\ X_1^2 = r\frac{\partial}{\partial x^1}. \tag{1}$$

However, the Dirac equation, with which [**5**] was concerned, is for electrons with spin. To introduce it we tensor the bundle with the four-dimensional trivial bundle on M. If σ is the representation of G to itself, then the equations are invariant under the transformation $F \to F'$ with

$$(1\otimes\sigma(g))F'(t,\vec{x}\ A(g)) = F(t,\vec{x}).$$

So we replace F by f with

$$f(t,r,g) = (1\otimes\sigma(g))F(t,r(0,0,1)A(g)) \tag{2}$$

to obtain equations invariant under right translations.

In terms of f the Dirac operator may be written

$$i\left(\gamma^0\frac{\partial}{\partial t} + \gamma^3\frac{\partial}{\partial r} + \frac{\gamma^1}{r}X^2 - \frac{\gamma^2}{r}X^1 + \frac{1}{r}\gamma^3\right) - m. \tag{3}$$

The notation is that of [**1**]. This need only be verified at $(t,r,1)$. Using equations (1) we see that the Dirac operator may be written as

$$i(1\otimes\sigma(g))\left(\gamma^0\frac{\partial}{\partial t} + \frac{\gamma^1}{r}X^2 - \frac{\gamma^2}{r}X^1 + \gamma^3\frac{\partial}{\partial r}\right)(1\otimes\sigma(g^{-1})) - m,$$

g being set equal to 1 after the differentiation. Thus it is

$$i\left(\gamma^0\frac{\partial}{\partial t} + \gamma^3\frac{\partial}{\partial r} + \frac{\gamma^1}{r}X^1 - \frac{\gamma^2}{r}X^1 - \frac{\gamma^1}{r}\sigma(X^2) + \frac{\gamma^2}{r}\sigma(X^1)\right) - m.$$

Since

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \gamma^j = \begin{pmatrix} 0 & \sigma^j \\ -\sigma^j & 0 \end{pmatrix} \quad (j=1,2,3)$$

and

$$\sigma(X^j) = \frac{i}{2}\begin{pmatrix} \sigma^j & 0 \\ 0 & \sigma^j \end{pmatrix},$$

the equation (3) follows.

The eigenfunction equation of the associated Hamiltonian is

$$\left\{\frac{1}{i}\begin{pmatrix} 0 & \sigma^3 \\ \sigma^3 & 0 \end{pmatrix}\frac{\partial}{\partial r} + \frac{1}{i}\begin{pmatrix} 0 & \sigma^1 \\ \sigma^1 & 0 \end{pmatrix}\frac{X^2}{r} - \frac{1}{i}\begin{pmatrix} 0 & \sigma^2 \\ \sigma^2 & 0 \end{pmatrix}\frac{X^1}{r} + \frac{1}{ir}\begin{pmatrix} 0 & \sigma^3 \\ \sigma^3 & 0 \end{pmatrix}\right\} f = \begin{pmatrix} E-m & 0 \\ 0 & E+m \end{pmatrix} f. \tag{4}$$

It cannot be interpreted until the formal expression on the left has been completely defined as an operator H that will be an extension of the closure H_0 of the obviously defined operator on functions with compact support on $\mathbf{R}^+ \times G$. Since H_0 will be seen not to be self-adjoint if $n \neq 0$, there will be some freedom in the choice of H.

To analyze H_0 and its self-adjoint extensions we can clearly consider the projections onto the space of functions f transforming according to a given representation of G. Take the representation of dimension $d+1$ acting on the polynomials of degree d in two variables with orthonormal basis $e_{j,k} = \binom{d}{j}^{1/2} x^j y^k, j+k=d$.

The entries of the column vector f will then be matrix coefficients of the representation and we can take them all from one column, the several columns entailing a multiplicity. As a consequence of (2) the first and the third entries will be multiples f_1/r and f_3/r of the matrix coefficient on the row with $j=k=n+1$, and the second and fourth entries will be multiples f_2/r and f_4/r of the coefficient in the row with $j-k=n-1$.

The denominator simplifies the inner product

$$\int_{\mathbf{R}^+ \times G} \|f\|^2 r^2 dr dg = \sum_{i=1}^{4} \int_0^\infty |f_i(r)|^2 dr$$

and, when we rewrite (4) in terms of the f_i, removes the last term on the left. We have

$$\sigma^1 X^2 - \sigma^2 X^1 = \begin{pmatrix} 0 & X^2 + iX^1 \\ X^2 - iX^1 & 0 \end{pmatrix}$$

and, in the Lie algebra,

$$X^2 + iX^1 = \frac{i}{2}(\sigma^2 + i\sigma^1) = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix},$$

$$X^2 - iX^1 = \frac{i}{2}(\sigma^2 - i\sigma^1) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Moreover

$$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} : e_{j,k} \to (j(k+1))^{1/2} e_{j-1,k+1},$$

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} : e_{j,k} \to ((j+1)k)^{1/2} e_{j+1,k-1}.$$

So the left side of (4) becomes

$$
\left\{\frac{1}{i}\begin{pmatrix}0&0&1&0\\0&0&0&-1\\1&0&0&0\\0&-1&0&0\end{pmatrix}\frac{\partial}{\partial r}+\frac{1}{ir}\begin{pmatrix}0&0&0&-\mu\\0&0&\mu&0\\0&-\mu&0&0\\\mu&0&0&0\end{pmatrix}\right\}\begin{pmatrix}f_1\\f_2\\f_3\\f_4\end{pmatrix} \tag{5}
$$

and the right side

$$
\begin{pmatrix}E-m&0&0&0\\0&E-m&0&0\\0&0&E+m&0\\0&0&0&E+m\end{pmatrix}\begin{pmatrix}f_1\\f_2\\f_3\\f_4\end{pmatrix}. \tag{6}
$$

Here $\mu=((j+1)k)^{1/2}=\sqrt{(d+1)^2-n^2/2}, j+k=d, j-k=n-1$. In the special case $d+1=-n$, the coefficients f_2 and f_4 are fictitious, $\mu=0$, and (5) becomes

$$
\frac{1}{i}\begin{pmatrix}0&1\\1&0\end{pmatrix}\frac{\partial}{\partial r}\begin{pmatrix}f_1\\f_3\end{pmatrix}. \tag{7}
$$

If $d+1=n$ then (5) becomes

$$
-\frac{1}{i}\begin{pmatrix}0&1\\1&0\end{pmatrix}\frac{\partial}{\partial r}\begin{pmatrix}f_2\\f_4\end{pmatrix}. \tag{8}
$$

Taking

$$
\begin{pmatrix}f_1\\f_2\\f_3\\f_4\end{pmatrix}=\begin{pmatrix}f\\\epsilon f\\g\\-\epsilon g\end{pmatrix}
$$

with $\epsilon=\pm1$, we decouple the system given by (5) and (6) into the systems

$$
\left\{\frac{1}{i}\begin{pmatrix}0&1\\1&0\end{pmatrix}\frac{\partial}{\partial r}+\frac{\epsilon\mu}{ir}\begin{pmatrix}0&1\\01&0\end{pmatrix}\right\}\begin{pmatrix}f\\g\end{pmatrix}=\begin{pmatrix}E-m&0\\0&E+m\end{pmatrix}\begin{pmatrix}f\\g\end{pmatrix}. \tag{9}
$$

In the two exceptional cases we have the system

$$
\pm\frac{1}{i}\begin{pmatrix}0&1\\1&0\end{pmatrix}\frac{\partial}{\partial r}\begin{pmatrix}f\\g\end{pmatrix}=\begin{pmatrix}E-m&0\\0&E+m\end{pmatrix}\begin{pmatrix}f\\g\end{pmatrix}. \tag{10}
$$

The operators appearing on the left of (9) and (10) have still to be defined. They are to be self-adjoint extensions of the closure G_0 of the obviously defined operators on smooth functions of compact support on $(0,\infty)$. We apply the theory of [**3**], XIII. 2, which obviously extends to systems. Denote the formal differential operator on the left of (9) or (10) by τ.

For equation (9) the roots of the indicial equation of τ at 0 are $\pm\mu$. Since $\mu \geq 1$, only one solution of

$$\tau \begin{pmatrix} f \\ g \end{pmatrix} = \pm i \begin{pmatrix} f \\ g \end{pmatrix} \tag{11}$$

is square-integrable on $(0, 1]$ and there are no boundary conditions ([**3**], XII.4.21, XIII.2.19, XIII.2.23). For equation (10) the indicial equation has the multiple root 0 at 0. So there are two independent boundary values. For both equations there is one square-integrable solution of (11) on $[1, \infty)$ and no boundary condition at ∞.

We conclude that for (9) the operator T_0 is already self-adjoint. It follows from Lemma XIII.4.23 of [**3**] and integration by parts that for (10)

$$-i(T_0^* F, F') + i(F, T_0^* F') = f\bar{g}' + g\bar{f}',$$

with $F = (f, g)^t, F' = (f', g')^t$. So the self-adjoint extensions T of T_0 are defined by $f(0) = i\lambda g(0), \lambda \in$ **R**, or $g(0) = 0$.

In all cases the discrete spectrum of

$$T + \begin{pmatrix} m & 0 & 0 & 0 \\ 0 & m & 0 & 0 \\ 0 & 0 & -m & 0 \\ 0 & 0 & 0 & -m \end{pmatrix}$$

is obtained by explicitly solving (9) or (10). For a square integrable solution E must be real. For (10) there is a square-integrable solution of the equation only if $m^2 > E^2$ and it is

$$f = e^{-r\sqrt{m^2 - E^2}}, \; g = \frac{\mp 1}{i} \sqrt{\frac{m - E}{m + E}} \; e^{-r\sqrt{m^2 - E^2}}.$$

It satisfies the boundary conditions if and only if $-\sqrt{(m + E)/(m - E)} = \lambda$. So there is a bound state if $-\infty < \lambda < 0$ and none otherwise.

For (9) there could be a solution square-integrable at ∞ only if $m^2 > E^2$ and then it would have to be a multiple of the pair with f equal to the Whittaker function $W_{0,\nu}(\alpha r)$ ([**7**]) where $\alpha = 2\sqrt{m^2 - E^3}, \nu = \mu - \frac{1}{2}, \epsilon > 0, \nu = \mu + \frac{1}{2}, \epsilon < 0$. This pair cannot be square-integrable near 0.

We can sum the discussion up with a theorem (cf. [**4**]).

Theorem. *If $n = 0$ the operator H_0 is self-adjoint. If $|n| > 0$ then there is a 1-parameter family of spherically symmetric self-adjoint extensions H of H_0, parametrized by $\lambda \in \mathbf{R} \cup \{\infty\}$ and no others. The operator*

$$H + \begin{pmatrix} m & 0 & 0 & 0 \\ 0 & m & 0 & 0 \\ 0 & 0 & -m & 0 \\ 0 & 0 & 0 & -m \end{pmatrix}$$

has a discrete spectrum if and only if $\lambda \in (-\infty, 0)$, and then it consists of the eigenvalue $E = m(\lambda^2 - 1)(1 + \lambda^2)^{-1}$ with multiplicity $|n|$.

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