

Visual Explanations

A review by Bill Casselman

Visual Explanations - Images and Quantities, Evidence and Narrative, Edward R. Tufte, Graphics Press, Cheshire, Connecticut.

Edward Tufte is a member of the political science department at Yale University whose work has apparently always had a mildly mathematical flavour. In 1975, while at Princeton University, he was asked to teach a seminar on statistics and statistical graphics. This seems to have been a turning point in his career. The eventual outcome was a series of three extremely attractive and intriguing books on what he calls information graphics—*The Visual Display of Quantitative Information* (1983), *Envisioning Information* (1990), and *Visual Explanations* (1997). Tufte was so concerned with quality and cost that he established his own press, which is dedicated exclusively to the publication of these books.

Pleasant books to look at, certainly. Not very expensive, considering the quality, which has been improving as new volumes appear, presumably because the endeavour has proven itself financially. And obviously not without relevance at least to some fields of applied mathematics and statistics, because much space in these books is spent discussing how to display large, complicated data sets. That was a major theme in the earlier volumes, and still plays a role in the most recent one. But there is another theme of interest to a wider range of mathematicians. It started off modestly earlier in the series, but has come to be more important in the most recent volume—*how to use illustrations to explain complicated, even abstract, ideas effectively*. For a mathematician, the most intriguing possibility that comes to mind is how to use illustrations in explaining mathematical proofs.

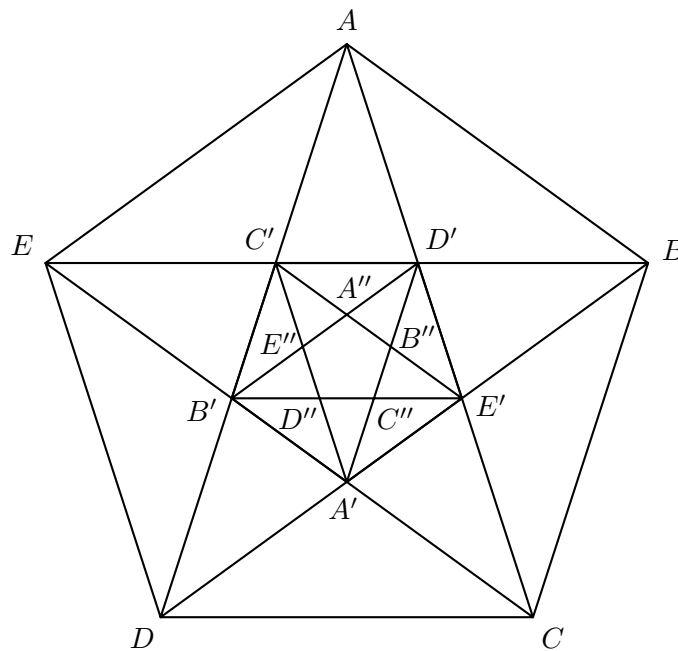
The first really striking example of this occurred in *Envisioning Information*, where Tufte spent several pages discussing Oliver Byrne's remarkable and appealing edition of Euclid's first six books. Briefly, what Byrne did in his edition of Euclid was to rewrite the arguments in pictures rather than words whenever possible. But there are other examples in Tufte's books, much simpler ones, taken from the mathematics literature—notably the pop-up tetrahedron from the Billingsley 1570 translation of Euclid into English (*Envisioning*, p. 16), the illustration from Descartes' *Principles of Philosophy* (*Explanations*, p. 61), Mosteller's advice on lecturing (*Explanations*, pp. 69–70), and the diagrams of light passing through a prism illustrating Joseph Lohne's observations on the corruption in time of Newton's originals (*Explanations*, p. 83). A more curious, even tantalizing, connection with mathematics is the title page of Margaret Norris' *The Notation of Movement* “with an Introduction by H. Levy, M.A., D.Sc., F.R.S.E., Professor of Mathematics, Imperial College of Science & Technology”. However, it is important to realize that much of what Tufte says about good graphic design in general applies to mathematics in particular, and that there are lessons here for all of us who incorporate figures in our work. I want to illustrate this by working out in detail, in one relatively simple example, how one might apply Tufte's principles. I want to make clear in advance that I have no pretensions to expertise here, but offer the following exercise as an experiment, to be criticized freely.

In the example, I want to show how graphics can be used to explain in an elementary way that the golden ratio is irrational. This is of course one of the oldest mathematical discoveries, and perhaps the first truly astonishing one. There is much discussion in the literature—almost entirely speculative, of course—as to how this irrationality was first found. One common and reasonable speculation is that it was arrived at by geometric reasoning—not that geometry provided at first

a completely rigorous proof in view of the then primitive state of Greek mathematics, but that it at least provided a convincing chain of reasoning of some kind leading to the result.

The golden ratio is also the ratio between the side and diagonal of a regular pentagon, and what we shall actually prove is that the side and diagonal of a regular pentagon are incommensurable. Very briefly, the idea of the argument used here is to see that if the side and diagonal are both multiples of an interval ϵ then so are the side and diagonal of the smaller pentagon at the center of the five-sided star whose vertices are those of the original pentagon. Recursion leads to a contradiction.

I will begin by quoting a very traditional approach to this question, from a 1945 paper by Kurt von Fritz on the discovery of incommensurability. Keep in mind throughout what is to follow that the point, as von Fritz says, is not merely to prove the result, but to make it “almost apparent at first sight”. Here is my copy of the figure drawn by von Fritz:



Von Fritz' Figure

Here is what he writes:

... the diameters of the pentagon form a new regular pentagon in the centre, ... the diameters of this smaller pentagon will again form a regular pentagon, and so on in an infinite process. It is ... easy to see that in the pentagons produced in this way $AE = AB'$ and $B'D = B'E'$ and therefore $AD - AE = B'E'$, and likewise $AE = ED' = EA'$ and $B'E' = B'D = B'E$ and therefore $AE - B'E' = B'A'$, and so forth ad infinitum, or, in other words, that the difference between the diameter and the side of the greater pentagon is equal to the diameter of the smaller pentagon, and the difference between the side of the greater pentagon and the diameter of the smaller pentagon is equal to the side of the smaller pentagon, and again the difference between the diameter of the smaller pentagon and its side is equal to to the diameter of the next smaller pentagon and so forth in infinitum. Since ever new regular pentagons are produced by the diameters it is then evident that the process

of mutual subtraction will go on forever, and therefore no greatest common measure of the diameter and the side of the regular pentagon can be found.

There is nothing wrong in the logic of this treatment, although it does stumble around a bit. What we are interested in right now, however, is how the argument relates to the figure. The answer, I think it is fair to say, is “not well”. Reading the original article is even more difficult than apparent here because, as often happens, the text and figure are on separate pages. What I claim is that *von Fritz stumbles precisely because he is trying to put in words what could have been far better put in pictures*. His one figure is not really used in a serious way, and essentially does no more than make the argument unambiguous. I also think it is fair to say that von Fritz is far from making the result apparent at first sight. Contrary to what he wants to do, he is preaching to the converted.

Of course one might object that a paper over 50 years old cannot be held completely responsible for its graphics, but actually von Fritz does better than most. At least one much more recent example (pp. 31–32 in Peter Cromwell’s otherwise admirable book) is, as far as this matter goes, worse.

Let’s see what help Tufte might be able to offer. The first step is to decide to take the graphics more seriously—to make the graphics the main part of the narrative, rather than subsidiary to it. The next step is to integrate text and graphics better; this is von Fritz’ major failing, because in reading his argument you are constantly forced to go back to the diagram, relocate yourself there, etc. A third is to determine which elements of the illustrations are important, and then to emphasize them. In von Fritz’ figure there are only the labelling of the vertices to orient the reader. But in fact the entities involved are not really the vertices at all, but instead various edges and sub-regions of the pentagon.

Perhaps the most succinct application of Tufte’s principles is found in Chapter 4 of *Visual Explanations*, ‘The Smallest Effective Difference’. It opens up with a diagram of the ear taken from the *Random House Dictionary of the English Language*, which has a remarkable resemblance to von Fritz’ diagram! Tufte redraws it to make it clearer, by carrying through the following ideas:

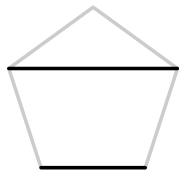
- Tone down the secondary elements of a picture in order to reduce visual clutter, to clarify the primary elements of the figure, and also to eliminate unwanted visual interactions. Tufte calls this **layering** the figure to produce a visual hierarchy.
- Reduce discontinuity in the exposition by replacing coded labels in the figure by direct ones. The general principle is to integrate text and graphics. One point is that unnecessary eye movements are fatal to easy comprehension.
- Produce emphasis by using the smallest possible effective distinctions. In practice this often, but not always, means replacing bold, strongly contrasting colors by quieter shades. This is perhaps the hardest of all sins to avoid, since it is often extremely tempting for the beginner to introduce strong colors when he can.

To this list might be added a few ideas from elsewhere in Tufte’s books:

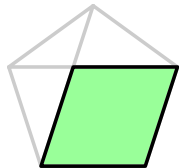
- Eliminate parts of the figure that do not actually add to its content.
- Use what Tufte calls **small multiples**, numerous repetitions of a single figure with slight variations.
- Make the graphics itself carry a story.

All of these are nearly self-evident principles, and if the use of graphics in mathematics were more sophisticated than it is now one might consider this an objection to Tufte’s books.

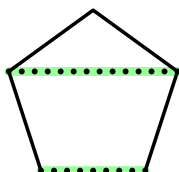
Here now is the argument I have made in up in an attempt to apply these principles:



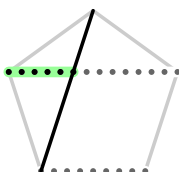
The basic fact is that in a regular pentagon a diagonal and the side opposite to it are parallel. This property in some sense characterizes the regular pentagon.



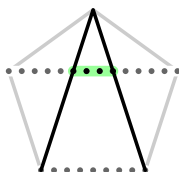
As a consequence, the shaded region shown at left is a parallelogram all of whose sides are equal (a rhombus).



Assume now that the side s and diagonal d in are commensurable, which is to say that they are both multiples of a common interval ϵ .



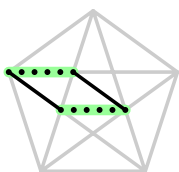
Then $d - s$ is also a multiple of the interval ϵ .



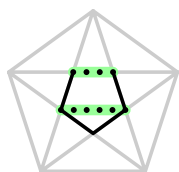
And so is the interval we get in the middle of the diagonal, which has length $d - 2(d - s) = 2s - d$.



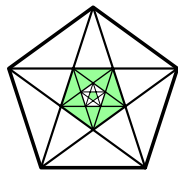
But this interval is the side of the smaller pentagon at the centre of the star we get by drawing all diagonals.



The figure emphasized in the diagram to the left is a parallelogram, since opposite sides are parallel to the same side of the pentagon. Therefore the quantity $d - s$ is the diagonal of the smaller pentagon.



Therefore under the assumption that the diagonal and side of a pentagon are multiples of a common interval ϵ , we deduce that so are the side and diagonal of the smaller pentagon inscribed in it.



We can reason in the same way about the pentagon in its interior in turn, etc. The interval ϵ will divide all the sides and diagonals of the infinite series of pentagons we get. But eventually the sides of those pentagons will be smaller than ϵ , a contradiction.

I imagine that some readers will find my argument distasteful. I am, however, in good company. It is no less than J. E. Littlewood who points out (p. 54 of the *Miscellany*)

A heavy warning used to be given that pictures are not rigorous; this has never had its bluff called and has permanently frightened its victims into playing for safety.

The validity of using pictures in proofs goes back to the origins of Greek mathematics and indeed to the origin of the word ‘graphics’ in a Greek word which, there is reason to believe, had the double meaning of ‘picture’ and ‘proof’ (Heath I, p. 203). In my experience criticism to the effect that pictures aren’t proofs usually means that the critic is accustomed to some kinds of picture proofs but not others. I have yet to see an explanation of Pythagoras’ Theorem, for example, without a picture—and the clearest proofs are certainly those which would be almost impossible to reduce to mere words. Such criticism also often means that the critic cannot imagine himself constructing the pictures accompanying the proof, but that is another problem—a question of time, energy, and skill rather than capability. As I have said elsewhere, it is unfortunate that the revolution in mathematical typesetting brought about by \TeX has not been matched by one in mathematical graphics. It is not easy to see how to make the right technology available to those who can use it, and how much work or even artistic talent one can reasonably expect from an author. Tufte says, in the www.amazon.com interview, about why he found himself in a new career, “I could both see and count. Usually those skills are not found together.” It may very well be true that many mathematicians are skillful at visualization, but this does not mean they can see what is in front of them.

Because of computers, we are now in an era where art and mathematics can collaborate in a way not seen since the Renaissance, and which distinguishes itself from the Renaissance in that it does not take the skills and imagination of Leonardo to do interesting and valuable work in this field. The opportunity of the moment is probably unique. It may not be long before exploration is replaced by settlement.

Final remarks about Tufte’s books. If you want exactly one of the three books, then *Visual Explanations* is probably the most useful, although *Envisioning* is arguably the most attractive. There is a noticeable redundancy among the books, which is mildly annoying. On the other hand, each of the three contains a few unique gems.

Technical remark. The pictures in this note were produced directly in PostScript. The advantage of doing this is that one has virtually complete control over the final product, albeit only with a fair amount of work. There are other possibilities—one very fine recent example of mathematical graphics at a high level is Needham’s *Complex Visual Analysis*, which used quite different tools.

References

Bill Casselman, *Geometry and PostScript*, <http://sunsite.ubc.ca/Euclid>.

Peter Cromwell, *Polyhedra*, Cambridge University Press, 1997.

Kurt von Fritz, ‘The discovery of incommensurability by Hippasus of Metapontum’, *Annals of Mathematics*, 1945, pp. 242–264.

Thomas Heath, *A History of Greek Mathematics*, Oxford Press, 1921.

J. E. Littlewood, *Littlewood's miscellany*, Cambridge University Press, 1988.

Tristan Needham, *Visual Complex Analysis*, Oxford University Press, 1997.

Also of interest are the interviews with Tufte on the Internet at

<http://www.amazon.com>

and

<http://www.ercb.com/feature/feature.0008.2.html>.