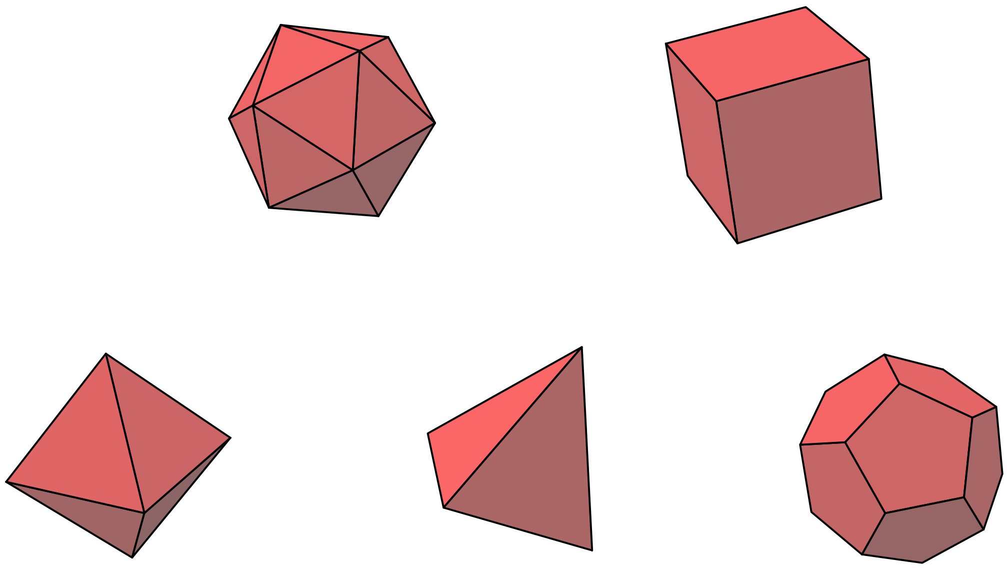
Chapter 11. The regular polyhedra

A regular figure is one which is . . . well, more *regular* than most. A polyhedron is a shape in three dimensions whose surface is a collection of flat polygons, and *a regular polyhedron is one all of whose faces and vertices look the same*. It has been known for a very long time that there are exactly five regular polyhedra.

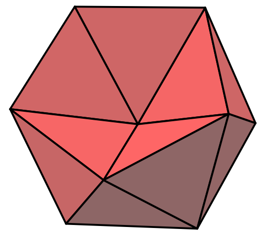


This is by no means a trivial fact, although it is one to which we have become accustomed over a period of at least 2300 years. The properties of regular polyhedra are in fact not easy to understand, and perhaps familiarity has made it more difficult to realize how remarkable they are. Although it is perhaps not the most mathematically sophisticated part of Euclid, the regular polyhedra are discussed only in the last book of the *Elements*, and the treatment is not at all transparent.

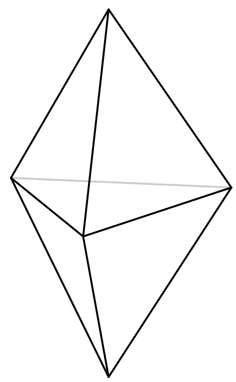
There are two quite different parts of the story here: (1) It is possible to construct five different regular polyhedra, and (2) it is not possible to construct any other types. You will appreciate later on what this distinction means. In both cases, at least to start with, I shall follow Euclid rather closely. At an elementary level, it is a hard act to beat. Later on, we shall look at contributions by Archimedes, Descartes, and even a few twentieth century mathematicians. I shall begin with part (2), and deal with the construction later on.

1. What exactly is a regular polyhedron?

It is important first to understand exactly what a regular polyhedron is. The first, and simplest condition, is that its faces, are to be regular polygons. Another is that all of these faces be congruent to each other. But there has to be some extra condition to guarantee regularity. For example, this condition of facial congruence will be satisfied by an icosahedron with some of its sides pushed in, which surely isn't in the same category as the other figures:



So there has to be an extra requirement. The obvious one is that the figure be **convex**, which means loosely that it bulges out. Technically this means that any two points in the figure can be joined by a segment contained completely inside the figure itself. But that isn't sufficient to characterize regularity either.



So we must impose the condition is all the vertices of the figure, as well as all its faces, 'look alike' in the sense that they are congruent. We shall in all impose these three conditions:

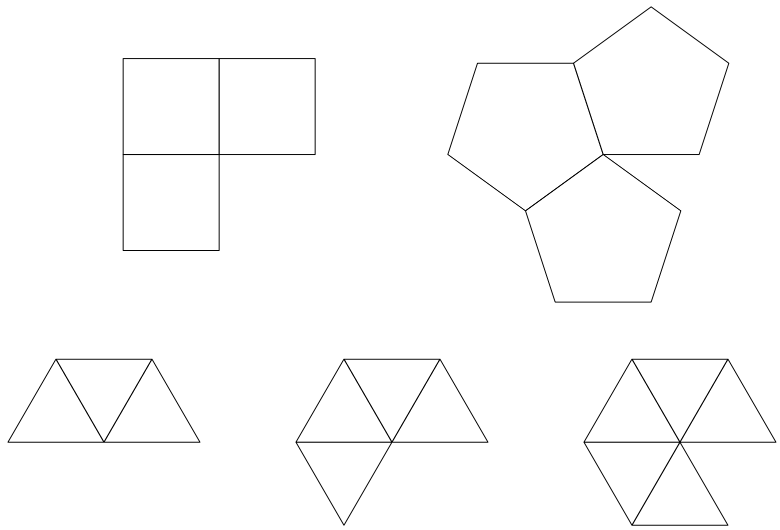
- *By definition, a regular polyhedron is one satisfying all three of these conditions:*
  - *(a) All of its faces are regular polygons.*
  - *(a) They are all congruent.*
  - *(b) The figure is convex.*
  - *(c) All of its vertices are congruent.*

In fact, these conditions are unnecesarily strong. It is actually the case that we need only require that the number of faces around each vertex be the same for all vertices. A remarkable theorem proven by the French mathematician Cauchy in the early nineteenth century asserts that this implies the much stronger third condition above. But it is not a simple result, and it is better in an elementary treatment not to depend on it.

2. There are no more than five regular solids

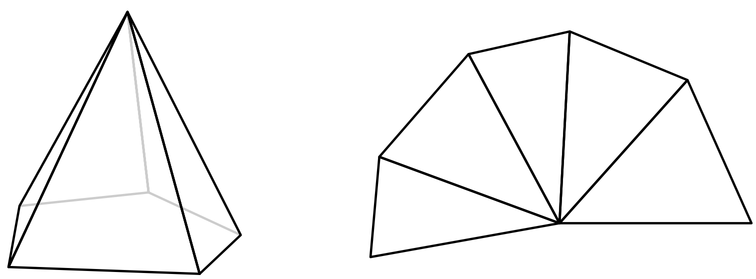
I shall first explain roughly why this is so, and then go over the argument later in detail.

This part of the argument just considers what happens around one of the vertices, call it P, of a regular polyhedron. Throw away all of the faces of the polyhedron which do not touch P. Then flatten out the faces that are left. For the polyhedra we know about, we get the following pictures of what I call the **splayed** vertices.



It is intuitively reasonably clear that when we do that, the vertex 'opens up' in the sense that in going around the vertex we don't go all around its image in the plane. In fact, this is a special case of Proposition XI.21 from Euclid, which is much more general:

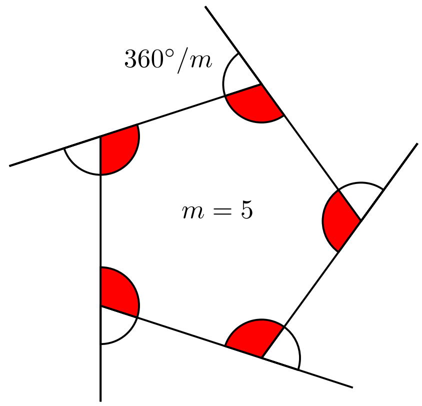
- *In going once around the faces touching any convex vertex, the angles we traverse always add up to less than* $360°$.



The word 'convex' here, as with the earlier use of the same word, means a vertex which always bulges out. Convexity is clearly a necessary condition, since if we are allowed to fold up the faces around a vertex like an accordion the proposition is no longer true. I shall come back later to give the details of the argument, nearly all of which arise in the very beginning (Book I) of Euclid. Let's assume for now that the result is true and see why it implies that there can be no more than five regular polyhedra.

Any regular polyhedron must have the property that all faces are congruent. This means that each one of them will be a regular polygon with the same number of sides. Furthermore, all the vertices are congruent, which means that the number of faces touching each vertex must be the same. Suppose that each face has m sides, and that each vertex is touched by n faces. What are the possibilities?

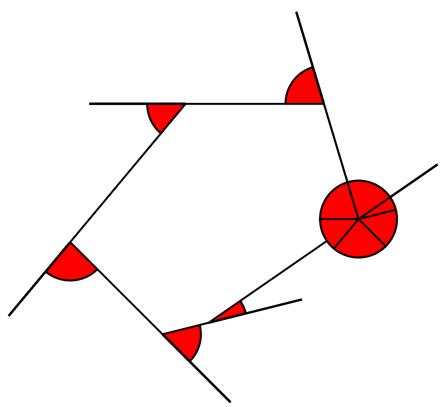
- *In a regular plane polygon of* m *sides, the angle at each corner is equal to* $180° - 360°/m$.



Since the inside angle at any corner is $180°$ less the angle turned at that corner, this follows immediately from a more intuitive result:

- *If we follow around the outside of a convex plane polygon, the total angle turned is* $360°$.

The proof of this is suggested immediately by the following diagram:



At any rate, if we have n polygons, each of m sides, at a convex vertex, then the total angle traversed as we go around the faces next to the vertex will be $n\,(180° - 360°/m)$ and this must be less than 360:

$$\begin{aligned} n(1 - 2/m)\,180° &< 360° \\ 180° - \frac{360°}{m} &< \frac{360°}{n} \\ \frac{1}{2} &< \frac{1}{m} + \frac{1}{n} \end{aligned}$$

We must have both m and n at least three, since our polyhedron must be a truly three-dimensional figure. Therefore

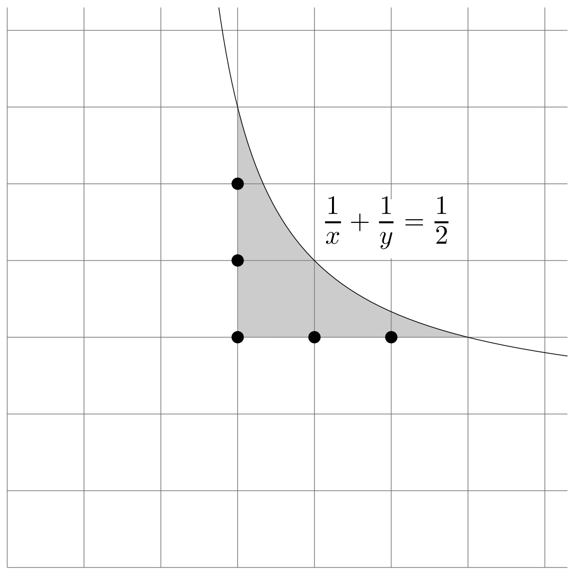
$$\frac{1}{2} < \frac{1}{m} + \frac{1}{n} \le \frac{1}{m} + \frac{1}{3}, \qquad \frac{1}{6} \le \frac{1}{n}, \qquad n \le 6\ .$$

Similarly, $m \le 6$. So we have only a finite number of possibilities to look at, examined in the following table, which shows $1/m + 1/n$ for $3 \le m, n \le 6$, except that those that don't qualify are left out:

m	n: 3	4	5	6
3	$2/3$	$7/12$	$8/15$	$-$
4	$7/12$	$-$	$-$	$-$
5	$8/15$	$-$	$-$	$-$
6	$-$	$-$	$-$	$-$

We see that there are exactly five possibilities, each corresponding to one of the known regular polyhedra.

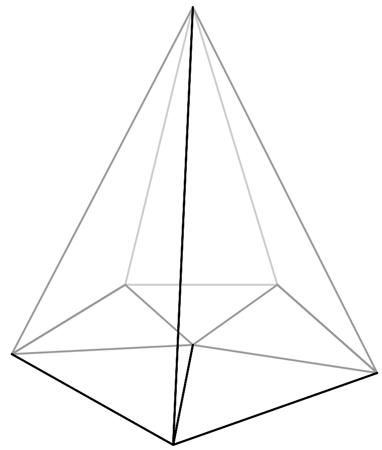
Another way to see which m and n qualify is to sketch the region $1/x+1/y > 1/2, x \geq 3, y \geq 3$ in the (x, y)-plane, and observe which points with integral coordinates lie inside it.



3. The proof of Euclid XI.21

The proof that Euclid gives for Proposition XI.21 involves a sequence of subsidiary results, mostly taken from Book I of the *The Elements*. Since the proposition itself seems, as do so many results in Euclid, almost obvious, I should say a few words of comment about the argument.

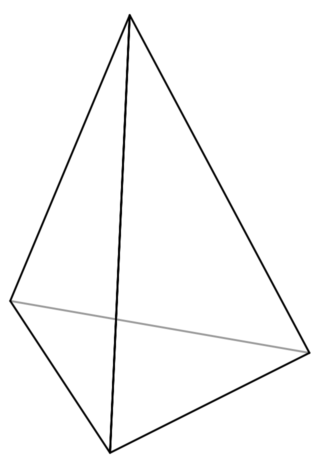
I think one's intuition that the Proposition is correct is based on the idea that projecting one of the facial angles of a convex vertex onto a plane spreads that angle out.



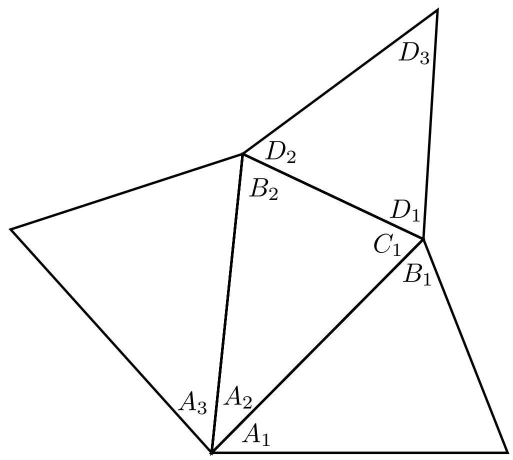
If this were true, then in order to prove Proposition XI.21 one could just project the 3D vertex into 2D, and compare the angles on the faces to those in the projection. Since the angles in the projection would be those around the projected vertex, they would add up to $360°$, and the proposition would follow immediately.

Unfortunately, the claim is false—angles in 3D often do project onto larger angles in 2D, but often onto smaller ones. In fact, if you think about it, it has to be that way because we can always project backwards as well! So it is not apparently true that we can make a direct comparison of the angles on each face with those in a 2D projection. Euclid must have been aware of this, although as usual he doesn't tell you more than you have to know. He manages, however, to get around the difficulties in a very elegant manner. I suppose his argument is a natural one, and one which some would perhaps call obvious, but nonetheless I believe it to be one of the highlights in *The Elements*.

I shall present the argument by a backwards progression. First of all, along with Euclid I shall assume that the vertex is surrounded by three faces, in order to make the reasoning a little more concrete.



We need to label the figure. Cut off the faces by a plane Π intersecting them transversely. Each face becomes a triangle, and the interior of Π cut off is also a triangle. To picture better what is going on, we can unfold and spread these triangles out on a plane. Label the angles in these triangles like this:



Here A_1, A_2, A_3 are the angles immediately surrounding the original vertex. Thus, each one of the lower vertices in this tetrahedron will have angles B, C, D around it. This is the crucial fact:

- *If B, C, and D are any three angles around a trihedral vertex, then $B + C > D$.*

I'll postpone the proof of this for a moment, but right now let's see why this implies Proposition XI.21. In each triangle, the sum of its interior angles must be $180°$. Therefore

$$\sum(A_i + B_i + C_i) = 3 \cdot 180° \ .$$

But in addition, according to the result we have yet to prove

$$\sum(A_i + B_i + C_i) > \sum A_i + \sum D_i, \qquad \sum A_i < \sum(A_i + B_i + C_i) - \sum D_i$$

since $B_i + C_i > D_i$. However

$$\sum D_i = 180°$$

since the D_i are all the interior angles of a triangle. This gives us

$$\sum A_i < \sum(A_i + B_i + C_i) - \sum D_i = 3 \cdot 180° - 180° = 360° \ ,$$

which is just what Proposition XI.21 asserts.

Suppose that, conversely, that one is given a collection of angles in the plane splayed out around a vertex, whose sum is less than $360°$. When can one construct a vertex in 3D that gives rise to it? Can one design an algorithm for doing this? First ofal, this is not always possible. For example, for three angles α, β and γ with $\alpha+\beta+\gamma < 360°$ it is possible only if $\alpha + \beta > \gamma$, for example (as we shall see later on). And if there are more than three angles the vertex will not be unique—it will not in fact be **rigid**. This means that if one is given such a vertex that one can always move the faces around as movable plates without changing their shape. This is just another way of saying that polygons in the plane are similarly flexible—for example, one can always deform a square into a rhombus.

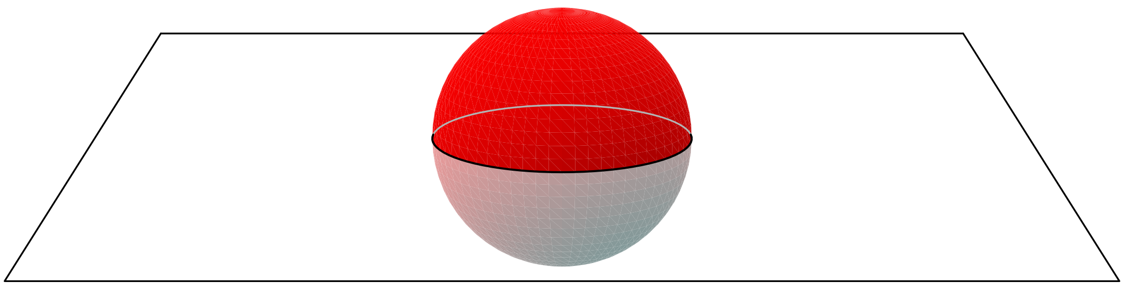
Exercise 3.1. *Suppose that $m\alpha < 360°$. Explain how to construct a regular vertex with vertex angle α—i.e. one whose orthogonal section is a regular polygon of m sides. (Hint: start with the regular polygon in a plane. The vertex should be somewhere on the perpendicular line through its centre. The fact that $m\alpha < 360°$ guarantees that the vertex can be located outide the plane.)*

4. Trihedral angles

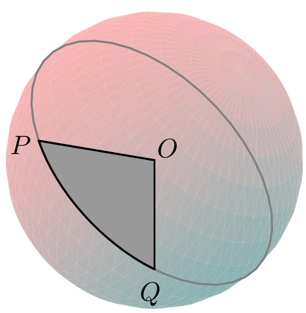
It remains to prove

- *If B, C, and D are any theee angles around a trihedral vertex, then $B + C > D$.*

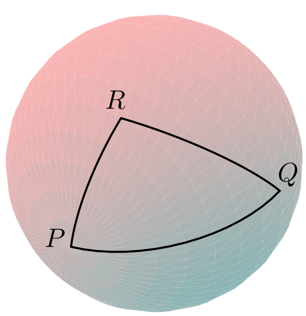
This is Proposition XI.20 of *The Elements*. Before proving it, I will make it somewhat more plausible by translating it into a statement about geometry on a sphere. A **great circle** on a sphere is the intersection of the sphere with a plane through its origin.



Between any two points P and Q on a sphere which are not directly opposite to each other there passes a unique great circle, that determined by the plane containing P and Q and the sphere's centre O. Distance on the sphere along a great circle is measured by the spanning angle POQ at the centre of the sphere.



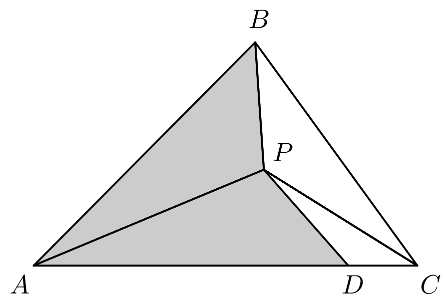
It is 'well known' that the arc of the great circle between two such points is the shortest route between them which lies entirely on the sphere. In particular, if R is a third point on the sphere which does not lie on the great circle arc between them, then the spherical distance PQ must be less than the sum of PR and RQ.



This is the 'spherical triangle inequality', analogous to the triangle inequality in the plane. Since spherical distances are proportional to central angles, it is equivalent to the claim we are trying to prove. So in effect, in proving the claim we are proving a well known fact about distances on a sphere.

How about the proof itself?

We follow Euclid. We start with the trihedral vertex P. If all three of the vertex angles are the same, the claim is trivial. So suppose that one is larger than another. Draw a line AB on the face with the smaller vertex angle, and then draw a line AC on the face with the larger one, with angles $PAB = PAC$. Here is a top view (still following Euclid):



The angle at the vertex opposite AC is by assumption greater than that opposite AB. The side AC must be longer than the side AB, by an early result from Book I of Euclid that I will recall in a moment. Therefore we can place a point D on AC making the triangles PAC and PAD congruent. By another well known result from Book I, the sum $AB + BC$ is greater than $AC = AD + DC$. Since $AD = AB$, $DC < BC$. By a third result from Book I, the angle DPC is less than the angle CPB. But then finally

$$APC = APD + DPC < APB + BPC\ .$$

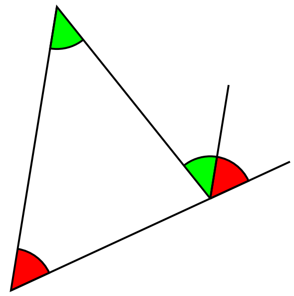
Here are the three results from Book I that we have used:

- *Sides opposite larger angles are longer.*
- *If we are given two triangles two of whose sides match in length, then the angle opposite the third side is larger in the triangle with the longer third side.*
- *In any triangle, the length of one side is less than the sum of the lengths of the two other sides.*

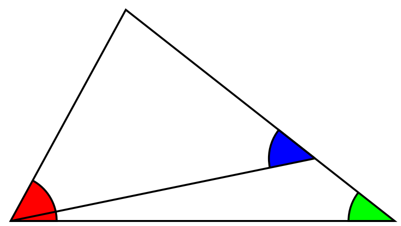
5. The results we need from Book I

There are three results we need from Book I of Euclid, and these in turn will take us back to others. Since we are not concerned here with complete rigour, but just with making the reasons as intuitively transparent as possible, the main difficulty is knowing where to begin. For many of these early results, I shall just exhibit pictures.

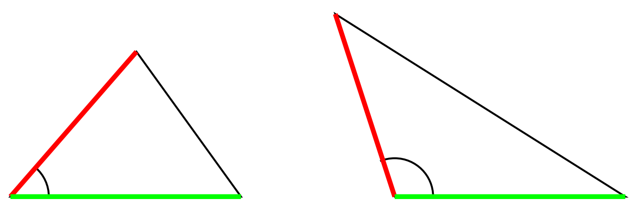
- *In any triangle, the exterior angle of one corner is equal to the sum of the opposite interior angles.*



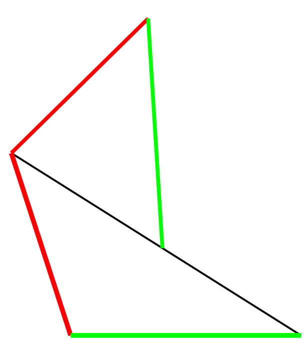
- *In any triangle, larger angles lie opposite longer sides.*



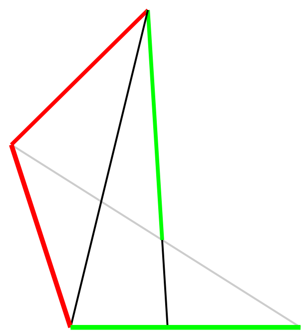
- *Given two triangles with two sides in each matching two sides in the other, the one with the longer third side has the larger angle opposite the third side.*



This we shall actually see proven. The demonstration I am about to give is attributed to Menelaus in Heath's comments on Proposition I.25. We start with the two triangles, pictured above. We translate the one with the smaller side and then rotate it and reflect it so as to get this picture:



Then we construct the isosceles triangle as shown below, and extend the line also. Finally, we apply the previous Proposition.



The last result to prove is the triangle inequality.

- *The length of any side of a triangle is less than the sum of the other two sides.*

I leave this as an exercise, including pictures. (It's Proposition I.20 in Euclid.)

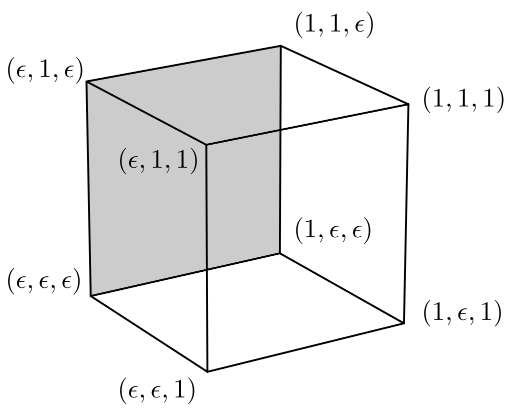
6. Constructing the regular polyhedra

It is important to understand that as far as showing that the regular solids can be constructed is concerned, the proof above is very limited in relevance. It says no more and no less than that *a single corner of each of the regular polyhedra can be constructed.* But constructing a corner is not the same as constructing the whole figure. It is not at all obvious that the construction of the corner can be extended to give the whole figure. Of course starting with one corner we can go on building new corners attached to what we already have, but there is no obvious reason why at some point we won't get some kind of peculiar incompatibility between pieces we have constructed. An argument which shows directly and uniformly in all cases that such an incompatibility never occurs was found, as far as I know, only very recently. The argument we shall see here looks at each case on its own. There is one notable feature, however—it turns out that four of the five can be constructed by starting with cubes!

In the rest of this section I shall describe without proof the essentials of construction in all cases. In the next I shall sketch the justification of the construction.

Cube

This is easy. I make its side of length 2, aligned along the axes, with one corner at $(-1, -1, -1)$. Then the corners are all points with either 1 or -1 as coordinate, making eight in all.



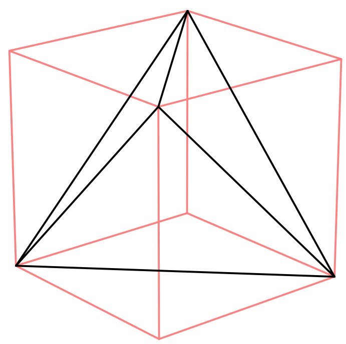
In the PostScript data file **`euclid.inc`** describing the regular polyhedra, these points are put into an array:

```
[[-1 -1 -1]
[-1 1 -1]
[1 1 -1]
[1 -1 -1]
[-1 -1 1]
[-1 1 1]
[1 1 1]
[1 -1 1]]
```

That is to say, I go around the back square with $z = -1$ in the positive orientation *as seen from behind*, starting from the origin, then around the front face $z = 1$ in a parallel track.

Tetrahedron

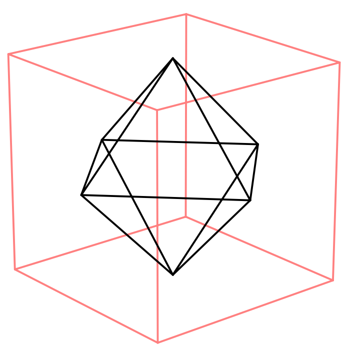
The vertices of the cube (x, y, z) with $x + y + z$ equal to -3 or 1 are the vertices of a regular tetrahedron, as are those where the sum is 3 or -1.



Exercise 6.1. *Prove this. Find an exact formula for the* **height** *of the tetrahedron, the distance from a vertex to the opposite face. Find the length of an edge.*

Octahedron

The centres of the faces of a cube form an octahedron.

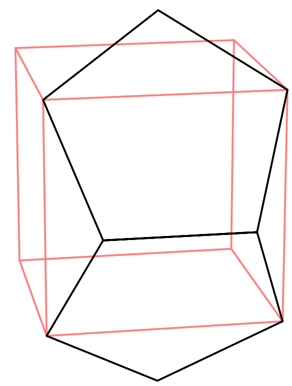


Exercise 6.2. *Find the length of an edge of this octahedron.*

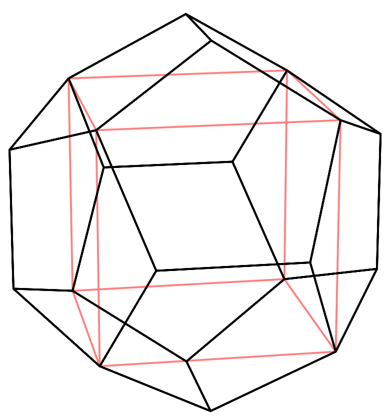
Dodeachedron

The tetrahedron and octahedron are relatively simple figures. It is perhaps more surprising that a dodecahedron can also be constructed by starting with a cube.

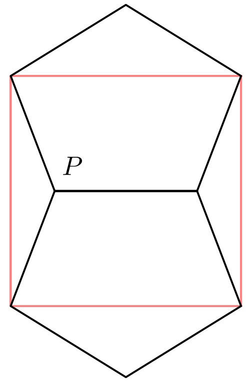
First construct a regular pentagon whose diagonal is equal to the side of the cube. Attach it along a diagonal to a side of the cube, in effect making the diagonal into a hinge. Attach another congruent pentagon to the opposite side. You can check that if the two pentagons lie flat on the common face of the cube, they will overlap. If they are rotated away from the cube, of course eventually they will have no intersection. So somewhere in between they can be situated like this, so they just touch:



The remarkable thing is that you can attach a pair of pentagons to each face of the cube in this way, changing the orientation if necessary, so as to have twelve pentagons making up a dodecahedron with the 12 pentagons for faces.

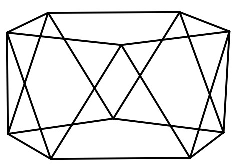


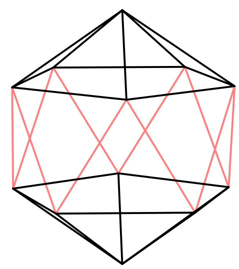
Exercise 6.3. *Find the coordinates (x, y, z) of the point P in 3D above the face of a cube making this work.*



Icosahedron

The icosahedron is different. Assemble a band of ten equilateral triangles, and then add to this two pentagonal caps of five equilateral triangles.





Exercise 6.4. *Find the coordinates of all the vertices, and in particular the vertical height of the top vertex and the top pentagon.*

7. Verifying regularity

I leave this as an exercise. The only serious problems are to show that the dodecahedron and icosahedron are completely regular, because this is very easy for the three simpler figures..

For the icosahedron, the faces join together and are all congruent by definition. What remains to be shown is that the vertices are congruent.

For the dodecahedron, in addition to showing that the vertices are all congruent, it must be shown that the pentagons constructed on each face actually attach to the pentagons from other faces in the way they should.

8. PostScript data for the regular polyhedra

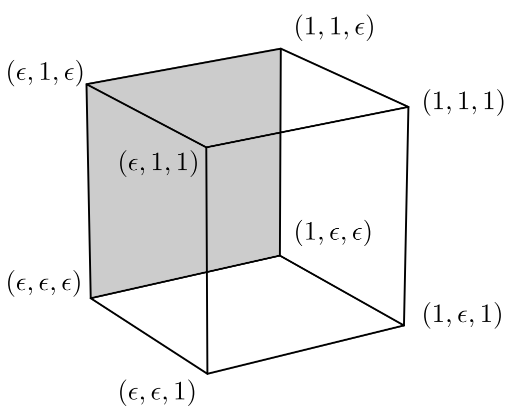
The regular polyhedra are catelaogued in the file **`euclid.inc`**. Their vertices are listed, and then their faces. Each face is an array of two items, first the coefficients **`[A B C D]`** such that $Ax + By + Cz + D \geq 0$ describes the outside of that face, and second the array of vertices on the face, traversed in a counter-clockwise direction.

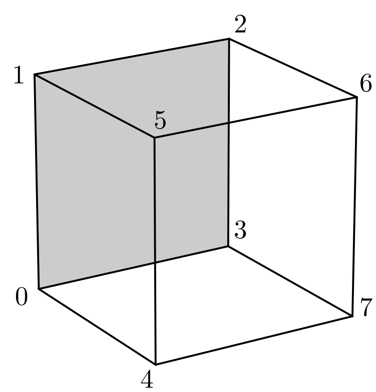
Here, for example, is the listing for the cube:

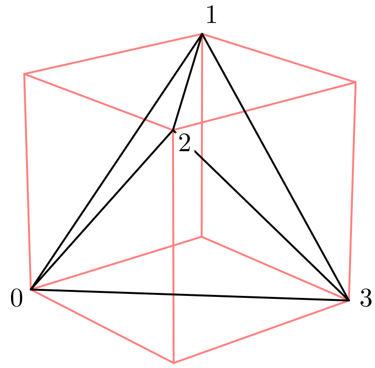
```
/cube-vertex [
[-1 -1 -1]
[-1 1 -1]
[1 1 -1]
[1 -1 -1]
[-1 -1 1]
[-1 1 1]
[1 1 1]
[1 -1 1]
] def

/cube-face [
[
  cube-vertex 0 get
  cube-vertex 1 get
  cube-vertex 2 get
  cube-vertex 3 get
]
[
  cube-vertex 4 get
  cube-vertex 7 get
  cube-vertex 6 get
  cube-vertex 5 get
]
[
  cube-vertex 0 get
  cube-vertex 4 get
  cube-vertex 5 get
  cube-vertex 1 get
]
[
  cube-vertex 6 get
  cube-vertex 7 get
  cube-vertex 3 get
  cube-vertex 2 get
]
[
  cube-vertex 2 get
  cube-vertex 1 get
  cube-vertex 5 get
  cube-vertex 6 get
]
[
  cube-vertex 0 get
  cube-vertex 3 get
  cube-vertex 7 get
  cube-vertex 4 get
]
] def

/cube [
8 dict begin
cube-face {
```

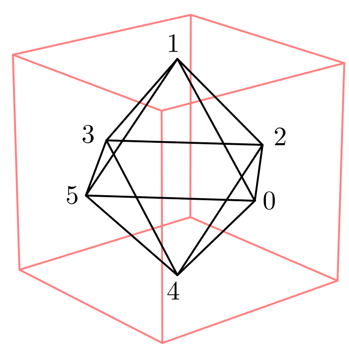
```
  % a face = an array normal plus vertex array
  [
    exch
    /f exch def
    f 1 get f 0 get vector-sub
    f 2 get f 1 get vector-sub
    cross-product normalized /n exch def
    [ n aload pop
    n f 0 get dot-product neg ]
    f
  ]
} forall
end
] def
```

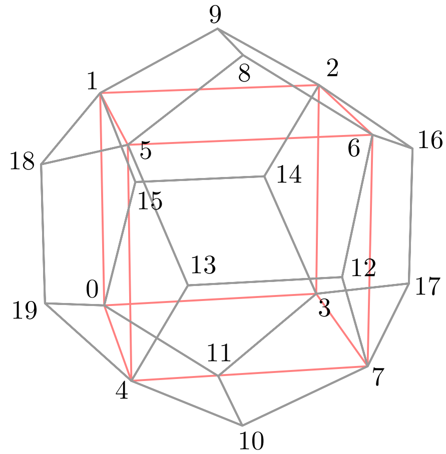
The file **`euclid.inc`** contains enough data to describe all the regular polyhedra. There are commands **`tetrahedron`**, **`octahedron`**, **`dodecahedron`**, and **`icosahedron`** which return for each figure an array of faces like the one shown above for the cube. In order to use it, you have to know the numbering scheme for the vertices. Here are some diagrams which do that. We start off with one we have seen before.











Finally, I just mention that the numbering of the icosahedron starts at the top and goes down.